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Integral theorems for functions and differential forms in C^m

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Introduction

I.1 The theory of holomorphic functions of several complex variables emerged as an attempt to generalize adequately onto the multidimensional situation the corresponding theory in one variable. In the course of a century long, extensive and intensive development it has proved to have beauty and profundity; many remarkable features and peculiarities have been found; new and far-reaching notions and concepts have been constructed. A multitude of applications to many areas of mathematics as well as to other sciences have been obtained.

I.2 At the same time, the deepening of the knowledge in several complex variables theory has been bringing those working in that field to the revelation of more and more paradoxical differences and distinctions between the structures of the two theories. S. Krantz, the author of many books and articles on several complex variables, writes in Preface of his book [Kr2, p.VII], that *"Chapter 0 consists of a long exposition of the differences between one and several complex variables."*

It is almost generally accepted that one of the deepest, most fundamental reasons for those differences lies in the absence of the universal and holomorphic Cauchy kernel i.e., a reproducing kernel which serves in any domain of $\mathbb{C}^m$, with reasonably smooth boundary but of any shape, and most importantly, is holomorphic. As S. Krantz writes on p.1 in [Kr2], *"there are infinitely many Cauchy integral formulas in several variables; nobody knows what the right one is, but there are several good candidates."*

In fact, what motivated us was exactly the desire to find *the right* Cauchy integral representation in several complex variables. To re-

alize what it really is, it proved to be necessary to come to a completely new approach: the *right* Cauchy integral representation can be constructed for a *right* set of functions which does not reduce to that of holomorphic functions but must be much more ample.

I.3 To explain the origin of the above-mentioned idea, let us analyze the basic elements which underlie one-dimensional, not multidimensional, complex analysis. There are many definitions of holomorphy there; all of them are equivalent, thus one can start from any of them. We shall use the standard notation:

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{I.3.1}$$

Null solutions to those operators provide us with the two classes of functions, respectively, holomorphic and antiholomorphic. Crucial is the fact that they factorize the two-dimensional Laplace operator $\Delta_{\mathbb{R}^2}$:

$$\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \circ \frac{\partial}{\partial z} = \frac{1}{4}\Delta_{\mathbb{R}^2}. \tag{I.3.2}$$

Combining this factorization with Green's (or the two-dimensional Stokes) formula, all the main integral theorems are routinely obtained: Cauchy and Morera, Borel-Pompeiu (= Cauchy-Green), Cauchy integral, etc.

As a matter of fact (although normally it is considered to be too trivial to mention), the definitions (I.3.1) and the factorization (I.3.2) are based on the excellent algebraic structure of $\mathbb{C}$, the range of functions under consideration. In particular, complex conjugation provides the possibility to factorize a non-negative quadratic form into a product of linear forms: $z \cdot \bar{z} = |z|^2 \geq 0$, and, of course, the factorization (I.3.2) is a manifestation of this property of complex numbers.

It is worthwhile to note that the commutativity of the multiplication in $\mathbb{C}$ is useful and pleasant to work with, but just in the above-mentioned integral theorems it is not of great importance.

I.4 Let $w = f(z) = u + iv,\ z = x + iy$, then the condition $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the system of the Cauchy-Riemann equations which

says that the components u, v of the holomorphic function f are not independent, but are interdependent. In other words one can say that the definition of holomorphy involves w and z *entirely*, wholly, not coordinate-wisely. This (trivial) observation will be helpful in realizing some essential aspects of what follows below.

I.5 Let now f be a holomorphic function in $\Omega \subset \mathbb{C}^m$, i.e., $\frac{\partial f}{\partial \bar{z}_1} = 0, \ldots, \frac{\partial f}{\partial \bar{z}_m} = 0$ in Ω, $m > 1$. Equivalently, there exist all complex **partial** derivates of the first order, with **no relations** between them. One sees immediately, hence, that the definition lacks the above described feature for $m = 1$: the definition includes certain conditions with respect to *each*, *partial* complex variable, z_k, and not with respect to the *entire* variable $z := (z_1, \ldots, z_m)$. Of course, this is related to the absence of two *mutually conjugate* operators factorizing the Laplace operator in $\mathbb{C}^m$. What is called *the Cauchy-Riemann conditions* in $\mathbb{C}^m$, should be more relevantly termed *partial Cauchy-Riemann conditions* to emphasize the difference in principle of both notions.

The idea of a holomorphic mapping loses much more from the original definition in $\mathbb{C}^1$. Indeed, if $\mathbf{F} = (f_1, \ldots, f_n)$ is a holomorphic mapping from $\Omega \subset \mathbb{C}^m$ into $\mathbb{C}^n$ then $\mathbf{F}$ keeps lacking any relation between complex partial derivatives of its components, and there are no relations, in general, between the components themselves.

I.6 Thus, looking for a *one-dimensional structure* in several complex variables we are going to depart from the following **heuristic reasonings**. Given a domain $\Omega \subset \mathbb{C}^m$, try to find the following objects:

1^0. A complex algebra $\mathcal{A}$ with unit, not necessarily commutative.

2^0. Two first-order partial differential operators with coefficients from $\mathcal{A}$, or from a wider algebra, denote them by $\mathfrak{D}$ and $\mathfrak{D}^*$, such that

$$\mathfrak{D} \circ \mathfrak{D}^* = \mathfrak{D}^* \circ \mathfrak{D} = \Delta_{\mathbb{C}^m}. \tag{I.6.1}$$

The idea of such a factorization is very well known in partial differential equations (see, e.g., [T1], [T2] but many other sources as well), and the fine point is contained, of course, in the last condition:

3^0. Holomorphic functions and mappings should belong to $\ker\mathfrak{D}$, or to $\ker\mathfrak{D}^*$.

To show that such a program is feasible is the aim of this book.

It is meant neither that in this setting the problem has a unique solution nor that the general case of arbitrary mappings will be covered. Our algebra $\mathcal{A}$ consists of 2×2 matrices whose entries are taken from the Grassmann algebra generated by differential forms with complex-conjugate differentials only, that is, of type $(0, q)$ in conventional terminology. Notice that it is possible to consider 1×2 columns instead of matrices, but then we loose the structure of a complex algebra in the range of functions, for which reason we chose to work with matrices.

I.7 The book is organized as follows. Chapter 1 recalls some basic notation which is necessary to work with functions and differential forms in $\mathbb{C}^m$. Chapter 2 introduces the main object of the study, differential forms whose coefficients are 2×2 matrices, as well as the differential operators acting on such differential forms and possessing the basic property (I.6.1).

The latter are called *the hyperholomorphic Cauchy-Riemann operators*. The fine point here is that their (2×2)-matrix coefficients contain not only differential forms but the so-called contraction operators also; the deep reasons for that will be explained in Chapter 9: a *right* algebra should be generated not only by differential forms.

As a matter of fact, the structure of the hyperholomorphic Cauchy-Riemann operators determines a special structure of other (2×2)-matrices involved — in particular, a unit normal vector to a surface in $\mathbb{C}^m$ is represented as such a matrix, the representation itself being an operator, not a differential form with matrix coefficients. The same about *the hyperholomorphic Cauchy kernel*, which is an operator, not a differential form, and which can be considered as a kind of a fundamental solution but in a specified meaning. All this leads to the hyperholomorphic versions of both the Stokes formula and the Borel-Pompeiu integral representation of a smooth differential form (here with (2×2)-matrix coefficients, of course), i.e., those versions

which are consistent with the hyperholomorphic Cauchy-Riemann operators. There is given a detailed analysis of the structure of the hyperholomorphic Borel-Pompeiu formula and of its intimate relation with the Bochner-Martinelli integral representation.

In Chapter 3, hyperholomorphic differential forms with (2×2)-matrix coefficients are introduced as null solutions of the hyperholomorphic Cauchy-Riemann operator. The class of such differential forms in a given domain includes both holomorphic and antiholomorphic functions (the latter considered as coefficients of specific differential forms), and all other *holomorphic-like* functions, i.e., those holomorphic with respect to certain variables and antiholomorphic with respect to the rest of them — all in the same domain and, again, taken as coefficients of specific differential forms. But this is not enough, and there are differential forms which do not correspond to any holomorphic-like functions. What is highly important here is the fact that just the whole class, not its more famous subclasses, preserves the deep similarity with the theory of holomorphic functions of one variable.

I.8 This similarity allows, in Chapters 4 through 7, to obtain quickly the main integral theorems. But even if, for instance, the Cauchy integral and the Morera theorems go in the usual way, anyhow certain peculiarities arise. The hyperholomorphic Cauchy-Riemann operator can be applied to a given matrix both on the left- and on the right-hand side.

There is no direct symmetry between left- and right-hand-side notions of hyperholomorphy, but we present versions of the Cauchy integral theorem and its inverse, the Morera theorem, which involves both types of hyperholomorphy.

The hyperholomorphic Cauchy integral formula (Chapter 6) represents any hyperholomorphic differential form as a surface integral with the hyperholomorphic Cauchy kernel. In particular, for holomorphic functions it reduces just to the Bochner-Martinelli integral representation of such functions which explains, in a certain sense, why the latter holds in spite of non-holomorphy of the Bochner-Martinelli kernel. One more manifestation of the above stated similarity is the solution of the non-homogeneous hyperholomorphic Cauchy-Riemann equation. In contrast to its counterpart for holo-

morphic Cauchy-Riemann equations, the hyperholomorphic case becomes trivial, since there exists a right inverse operator for the hyperholomorphic Cauchy-Riemann operator. All this is rigorously analyzed in Chapter 7, where many interpretations are also given, but the most remarkable applications are moved to the next Chapter.

I.9 In Chapter 8, differential forms are considered which are, simultaneously, $\overline{\partial}$-closed and $\overline{\partial}^*$-closed. They form a subclass of hyperholomorphic differential forms, but they are of independent interest and of importance from the point of view of conventional multidimensional complex analysis. That is why we, first of all, describe the direct corollaries of the theorems which have been proved for general hyperholomorphic differential forms. What is more, there are several results here which may be viewed also as corollaries, being at the same time much less direct and evident. One of them concerns the $\overline{\partial}$-problem for functions and differential forms in an **arbitrary**, i.e., of an arbitrary shape, domain in $\mathbb{C}^m$ with a piecewise smooth boundary. There is given a necessary and sufficient condition on the given $(0,1)$-differential form g in order for the equation $\overline{\partial} f = g$ to have a solution which is a function. The condition is quite explicit and verifiable: a $(0,2)$-differential form whose coefficients are certain improper integrals of g should satisfy the complex Hodge-Dolbeault system, i.e., should be $\overline{\partial}$-closed and $\overline{\partial}^*$-closed. A particular solution is again quite explicit, being a sum of improper integrals of the same type as above. If g is an **arbitrary** differential form (with smooth coefficients) then for the problem $\overline{\partial} f = g$ the necessary and sufficient condition obtained is not that explicit, but the particular solution has the same transparent structure as the one described above.

There exists a huge amount of literature on the $\overline{\partial}$-problem, see, e.g., [AiYu], [Ko], [Li], [Ky], [R], [Kr1], [Kr2], but in no way do we pretend that the above list is complete or even representative. It is a separate task to compare what has been obtained already on the $\overline{\partial}$-problem with the approach of this book.

I.10 In the same Chapter 8, we establish also a deep relation between solutions of the complex Hodge-Dolbeault system and the

Koppelman formula. The latter one is a representation of a smooth $(0,2)$-differential form as a sum of a surface integral and of two volume integrals. For the case of functions, i.e., of $(0,0)$-differential forms, the volume integrals disappear on holomorphic functions, and thus it is important to have a class of differential forms on which the volume integrals in the Koppelman formula disappear also. We show that the Koppelman formula is a particular case of the hyperholomorphic Borel-Pompeiu integral representation, which leads immediately to the conclusion that the volume integrals in the Koppelman formula are annihilated by the solutions of the complex Hodge-Dolbeault system.

We believe this will have deep repercussions for the theory of complex differential forms.

I.11 Although all the eight first chapters are written in the language of complex analysis, the underlying ideas were inspired by the authors' experience in research in Clifford and quaternionic analysis. What is the direct relation between those, at the present time, formally different areas of analysis is explained in Chapter 9. It appears that the hyperholomorphic theory restricted onto (2×2) matrices with equal rows is isomorphic to the function theory for the Dirac operator of Clifford analysis, see the books [BrDeSo], [DeSoSo], [Mit], [KrSh], [GüSp1], [GüSp2], [GiMu]. But we refer to many articles as well; other important aspects of the Dirac operators one can find in [BeGeVe] for instance. The general case of (2×2) matrices does not reduce to the theory of **one** Dirac operator but is a kind of a direct sum of the theories for **two** Dirac-like operators considered in the same domain of $\mathbb{C}^m$. The peculiarity of this relation is the necessity to use not the canonical basis of the Clifford algebra but the so-called Witt basis which fits perfectly well into the complex analysis setting. What is more, one half of the elements of the Witt basis generates the algebra of elementary differential forms while the other half generates the contraction operators. Hence the function theory using only differential forms lacks the *symmetry* of Clifford analysis, which causes new phenomena, such as, for instance, the fact that the hyperholomorphic Cauchy kernel is an operator, not a differential form.

I.12 Only small fragments of the book have been published already [RSS2], [RSS3], but the joint article by the authors [RSS1] may be considered as directly antecedent to the book; what is more, it may be seen as a direct impulse to realizing certain important ideas of it. At the same time, in their preceding separate works one can find many observations, hints, and indications on the relations between several complex variables theory and Clifford analysis ideas: F. Sommen treated those relations in [So1] (considering integral transform between monogenic functions of Clifford analysis and holomorphic functions of several complex variables), [So2] (deriving the Bochner-Martinelli formula), [So3]–[So5], see also the books [BrDeSo] and [DeSoSo]; M. Shapiro treated the applications of quaternionic analysis to holomorphic functions in $\mathbb{C}^2$ in joint papers with N. Vasilevski [VaSh1], [VaSh2], [VaSh3] and with I. Mitelman [MiSh1], [MiSh2]; see also the paper [Sh1]; the papers by M. Shapiro [Sh2] and by R. Rocha-Chávez and M. Shapiro [RoSh1], [RoSh2] do not have any direct relation to several complex variables, but they contain several important ideas which were very helpful in realizing some essential aspects of the book.

We know of not too many other papers on the topic. J. Ryan in [Ry1], [Ry2] considered a subclass of holomorphic functions for which a function theory is valid with the structure quite similar to that of Clifford analysis. V. Baikov [Ba] and V. Vinogradov [Vi] considered boundary value properties of holomorphic functions in, respectively, $\mathbb{C}^2$ and $\mathbb{C}^m$ using ideas from quaternionic and Clifford analysis. Quite recently S. Bernstein [Be] and G. Kaiser [Ka] found new connections between holomorphic functions and Clifford analysis.

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Chapter 1

Differential forms

1.1 Usual notation

We shall denote by $\mathbb{C}$ the field of complex numbers, and by $\mathbb{C}^m$ the m-dimensional complex Euclidean space. If $z \in \mathbb{C}^m$, then by $z_1, \ldots,$ z_m we denote the canonical complex coordinates of z. For $z,\ z' \in \mathbb{C}^m$ we write:

$$\begin{aligned} \bar{z} &:= (\bar{z}_1,\ \ldots,\ \bar{z}_m)\,, \\ \langle z,\ z' \rangle &:= z_1 z'_1 + \cdots + z_m z'_m, \\ |z| &:= \left(|z_1|^2 + \cdots + |z_m|^2 \right)^{\frac{1}{2}} = \sqrt{\langle z,\ \bar{z} \rangle}. \end{aligned}$$

$\mathbb{R}$ denotes the field of real numbers, and $\mathbb{R}^m$ denotes the m-dimensional real Euclidean space.

Topology in $\mathbb{C}^m$ is determined by the metric $d\,(z,\ z') := |z - z'|$.

Orientation on $\mathbb{C}^m$ is defined by the order of coordinates $(z_1,\ \ldots,$ $z_m)$, which means that the differential form of volume is

$$dV := (-1)^{\frac{m(m-1)}{2}} \frac{(-1)^m}{(2i)^m} dz \wedge d\bar{z} = (-1)^{\frac{m(m-1)}{2}} \frac{1}{(2i)^m} d\bar{z} \wedge dz,$$

where

$$\begin{aligned} dz &:= dz^1 \wedge \ldots \wedge dz^m, \\ d\bar{z} &:= d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^m. \end{aligned}$$

If $z \in \mathbb{C}^m$ then

$$\begin{aligned} x_j &:= Re\,(z_j) \in \mathbb{R}, \\ y_j &:= Im\,(z_j) \in \mathbb{R}. \end{aligned}$$

So, one can write $z = (x_1 + y_1 \cdot i,\ \ldots,\ x_m + y_m \cdot i)$. Hence $\mathbb{C}^m \cong \mathbb{R}^{2m}$ as oriented real Euclidean spaces, where the orientation in $\mathbb{R}^{2m}$ is defined by the order of coordinates $(x_1,\ y_1,\ \ldots,\ x_m,\ y_m)$, which means that the differential form of volume on $\mathbb{R}^{2m}$ is $dx^1 \wedge dy^1 \wedge \ldots \wedge dx^m \wedge dy^m$.

The word *domain* means an **arbitrary** (not necessarily connected) open set. The word *neighborhood* means an **open** neighborhood.

Some more standard notations:

1. $\mathbb{N}$ denotes the set of all positive integers,
2. $\mathbb{B}\,(z;\ \varepsilon) := \{\zeta \in \mathbb{C}^m \mid |z - \zeta| < \varepsilon\}$,
3. $\mathbb{S}\,(z;\ \varepsilon) := \{\zeta \in \mathbb{C}^m \mid |z - \zeta| = \varepsilon\}$,
4. $E_{2\times 2} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
5. $\breve{E}_{2\times 2} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Mention that $\breve{E}^2_{2\times 2} = E_{2\times 2}$.

1.2 Complex differential forms

The term **"differential form"** (or simply **"form"** and **d.f.** sometimes) will be used for differential forms with measurable complex-valued coefficients. The support of a differential form F will be denoted by $supp\,(F)$. For a fixed $k \in \mathbb{N}$, C^k-forms are those forms with k times continuously differentiable coefficients (this definition is independent of the local coordinate system of class C^{k+1}). Continuous forms will be called also C^0-forms, and $F \in C^\infty$ means that F is a form of class C^k for any $k \in \mathbb{N}$.

A form F of class C^k defined on $\mathbb{C}^m$ is called an $(r,\ s)$-form (i.e., a form of bidegree $(r,\ s)$) if, with respect to local coordinates $(z_1,\ \ldots,\ z_m)$ of class C^{k+1}, $0 \leq k \leq \infty$, it is represented as

$$F(z) = \sum_{|\mathbf{j}|=r,\ |\mathbf{k}|=s} F_{\mathbf{jk}}(z)\, dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \tag{1.2.1}$$

where the summation runs over all strictly increasing r-tuples $\mathbf{j} = (j_1,\ \ldots,\ j_r)$ and all strictly increasing s-tuples $\mathbf{k} = (k_1,\ \ldots,\ k_s)$ in $\{1,\ \ldots,\ m\}$, and $dz^{\mathbf{j}} := dz^{j_1} \wedge \ldots \wedge dz^{j_r}$, $d\bar{z}^{\mathbf{k}} := d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_s}$, with the coefficients $F_{\mathbf{jk}}$ being complex-valued functions of class C^k.

It is worthwhile to note that although we use the same letter z both for independent variable and for differentials dz^q, $d\bar{z}^p$, it is sometimes convenient and necessary to distinguish between them, so we will write $d\zeta^q$, $d\bar{\zeta}^p$ or dw^q, $d\bar{w}^p$, etc. This causes no abuse of notation, because these differentials do not depend on z. In that occasion, we will write $F\left(z,\ d\zeta,\ d\bar{\zeta}\right)$ instead of $F(z)$.

1.3 Operations on complex differential forms

Consider the following important differential operators. The linear contraction operators $\widehat{d\bar{z}^q}$ and $\widehat{dz^q}$ are defined as endomorphisms by their action on the generators:

1. if $q = k_p$, then

$$\widehat{d\bar{z}^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{d\bar{z}^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} :=$$

$$:= (-1)^{|\mathbf{j}|+p-1}\, dz^{\mathbf{j}} \wedge d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_{p-1}} \wedge \bar{z}^{k_{p+1}} \wedge \ldots \wedge d\bar{z}^{k_s},$$

2. if $q \notin \{k_1,\ \ldots,\ k_s\}$, then

$$\widehat{d\bar{z}^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{d\bar{z}^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} := 0,$$

3. if $q = j_p$, then

$$\widehat{dz^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{dz^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} :=$$

$$:= (-1)^{p-1}\, dz^{j_1} \wedge \ldots \wedge z^{j_{p-1}} \wedge dz^{j_{p+1}} \wedge \ldots \wedge dz^{j_r} \wedge d\bar{z}^{\mathbf{k}},$$

4. if $q \notin \{j_1, \ldots, j_r\}$, then

$$\widehat{dz^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{dz^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} := 0.$$

Now for F of the form

$$F := \sum_{\mathbf{j},\ \mathbf{k}} F_{\mathbf{jk}} dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \tag{1.3.1}$$

with $\{F_{\mathbf{jk}}\} \subset C^1\left(M \subset \mathbb{C}^m;\ \mathbb{C}\right)$, we set, as usual,

$$\begin{aligned}
\bar{\partial}\left[F\right] &:= \sum_{\mathbf{j},\ \mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial \bar{z}_q} d\bar{z}^q \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \\
\partial\left[F\right] &:= \sum_{\mathbf{j},\ \mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial z_q} dz^q \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \\
\bar{\partial}^*\left[F\right] &:= \sum_{\mathbf{j},\ \mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial z_q} \widehat{d\bar{z}^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] = \\
&= \sum_{\mathbf{j},\ \mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial z_q} \widehat{d\bar{z}^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \\
\partial^*\left[F\right] &:= \sum_{\mathbf{j},\ \mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial \bar{z}_q} \widehat{dz^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] = \\
&= \sum_{\mathbf{j},\ \mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial \bar{z}_q} \widehat{dz^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \\
d\left[F\right] &:= \bar{\partial}\left[F\right] + \partial\left[F\right]
\end{aligned}$$

(these definitions are independent of the local coordinate system of class C^2), where

$$\frac{\partial}{\partial \bar{z}_q} := \frac{1}{2}\left(\frac{\partial}{\partial x_q} + i\frac{\partial}{\partial y_q}\right), \qquad \frac{\partial}{\partial z_q} := \frac{1}{2}\left(\frac{\partial}{\partial x_q} - i\frac{\partial}{\partial y_q}\right).$$

Observe that $\widehat{d\bar{z}^q}$ only looks like a differential form but it is not; it is an endomorphism, so its wedge multiplication does not possess

all usual properties, and one should be careful working with such products.

Anyhow, with the above agreement, the differential form $\bar{\partial}[F]$ can be interpreted as a specific *exterior product* of a differential form F with the *differential form whose coefficients are partial derivations (not partial derivatives of a function)*, i.e., with $\bar{\partial} := \sum\limits_{q=1}^{m} \frac{\partial}{\partial \bar{z}_q} d\bar{z}_q$; what is more, in this sense F is multiplied by $\bar{\partial}$ on the left-hand side:

$$\bar{\partial} \wedge F := \bar{\partial}[F]. \tag{1.3.2}$$

Of course, it is assumed here that a scalar-valued function commutes with basis differentials. The same interpretations are valid for all other operations introduced above.

This observation is heuristically relevant, since it leads to the question, is it worthwhile to change *the order of multiplication* in (1.3.2)? We define now

$$\begin{aligned}
\bar{\partial}_r[F] &:= F \wedge \bar{\partial} := \sum_{\mathbf{j},\,\mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial \bar{z}_q} dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} \wedge d\bar{z}^{q}, \\
\partial_r[F] &:= F \wedge \partial := \sum_{\mathbf{j},\,\mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial z_q} dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} \wedge dz^{q}, \\
\bar{\partial}^*_r[F] &:= F \wedge \bar{\partial}^* := \sum_{\mathbf{j},\,\mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial z_q} dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} \wedge \widehat{d\bar{z}^{q}}, \\
\partial^*_r[F] &:= F \wedge \partial^* := \sum_{\mathbf{j},\,\mathbf{k}} \sum_{q=1}^{m} \frac{\partial F_{\mathbf{jk}}}{\partial \bar{z}_q} dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} \wedge \widehat{dz^{q}}, \\
d_r[F] &:= F \wedge d := \bar{\partial}_r[F] + \partial_r[F]
\end{aligned}$$

(this definition is independent of the local coordinate system of class C^2). Note that in contrast with the first two formulas (which lead to differential forms as a result), the next two formulas give operators acting on differential forms. So when we simultaneously use both $\bar{\partial}_r[F]$, $\partial_r[F]$ and $\bar{\partial}^*_r[F]$, $\partial^*_r[F]$, then we identify $\bar{\partial}_r[F]$, $\partial_r[F]$ with operators of multiplication by them. Notice that we will not use the

notation like $F \wedge \bar{\partial}$ to avoid possible confusion, in particular since $F \wedge \bar{\partial}$ can be seen as $F \circ \bar{\partial}$.

Note that for all $(r,\ s)$-form F of class C^1,

$$\bar{\partial}_r\,[F] = (-1)^{r+s}\,\bar{\partial} \wedge F := (-1)^{r+s}\,\bar{\partial}\,[F]\,.$$

This apparently insignificant difference will become essential later.

Let

$$\Delta_{\mathbb{C}^m} := \sum_{k=1}^{m} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} = \frac{1}{4} \sum_{k=1}^{m} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right) = \frac{1}{4} \Delta_{\mathbb{R}^{2m}}$$

be the complex Laplace operator in $\mathbb{C}^m$ whose action on a differential form of class C^2 is defined naturally to be component-wise: for F being as in (1.3.1) we put

$$\Delta_{\mathbb{C}^m}\,[F] := \sum_{\mathbf{j},\ \mathbf{k}} \Delta_{\mathbb{C}^m}\,[F_{\mathbf{jk}}]\,dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}. \tag{1.3.3}$$

Then, the following operator equalities hold on differential forms of class C^2:

$$\begin{aligned} \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} &= \Delta_{\mathbb{C}^m}\,, \\ \partial\partial^* + \partial^*\partial &= \Delta_{\mathbb{C}^m}\,. \end{aligned} \tag{1.3.4}$$

they are of extreme importance for the whole theory.

1.4 Integration with respect to a part of variables

Let X, Y be real manifolds of class C^1, and let F be a differential form on $X \times Y$. Let $\mu = dim_{\mathbb{R}}\ X$, $\nu = dim_{\mathbb{R}}\ Y$, let $(z_1,\ \ldots,\ z_\mu)$ be local coordinates of class C^1 in some open $V \subset X$ and let $(\zeta_1,\ \ldots,\ \zeta_\nu)$ be local coordinates of class C^1 in some open $U \subset Y$. Consider the unique representation

$$F\,(z,\ \zeta) = \sum_{\alpha} F_\alpha\,(z,\ \zeta) \wedge d\zeta^\alpha,$$

where α runs over all strictly increasing r-tuples $\alpha = (\alpha_1,\ \ldots,\ \alpha_r)$ in $\{1,\ \ldots,\ \nu\}$ with $0 \le r \le \nu$, $d\zeta^\alpha := d\zeta^{\alpha_1} \wedge \ldots \wedge d\zeta^{\alpha_r}$, and $F_\alpha = F_\alpha\,(\cdot,\ \zeta)$ is a family of differential forms on X which depends on

$\zeta \in U$. If X is oriented and the integrals $\int\limits_V F_\alpha(z,\,\zeta)$ exist for all fixed $\zeta \in U$ and any strictly increasing r-tuple α in $\{1,\,\ldots,\,\nu\}$ with $0 \leq r \leq \nu$, then

$$\int\limits_V F(z,\,\zeta) := \sum_\alpha \left(\int\limits_V F_\alpha(z,\,\zeta) \right) d\zeta^\alpha, \qquad \zeta \in U, \tag{1.4.1}$$

where α runs over all strictly increasing r-tuples in $\{1,\,\ldots,\,\nu\}$, with $r = 0,\,\ldots,\,\nu$. The result of this integration is a differential form on U, that is independent of the choice of the local coordinates $(\zeta_1,\,\ldots,\,\zeta_\nu)$. Therefore $\int\limits_V F(z,\,\zeta)$ is well-defined for all $\zeta \in U$. Notice that this definition implies that $\int\limits_V F_\alpha(z,\,\zeta) = 0$ if F_α does not contain monomials which are of degree μ in X.

1.5 The differential form $|F|$

For an oriented real manifold X of class C^1 with $dim_{\mathbb{R}}\, X = \mu$ and for a differential form F on X, the differential form $|F|$ is defined as follows: if $(x_1,\,\ldots,\,x_\mu)$ are positively oriented coordinates of class C^1 in some open set $U \subset X$ and $F = F_{1\ldots\mu} dx^1 \wedge \ldots \wedge dx^\mu$ on U (with $F_{1\ldots\mu}$ a complex-valued function), then

$$|F| := |F_{1\ldots\mu}|\, dx^1 \wedge \ldots \wedge dx^\mu \qquad \text{on } U. \tag{1.5.1}$$

If $|F|$ is integrable then F is also integrable and

$$\left| \int\limits_X F \right| \leq \int\limits_X |F|\,.$$

Given two μ-forms F, G on X, we write

$$|G| \leq |F|$$

if for their respective representations (1.5.1) one has on U:

$$|G_{1\ldots\mu}| \leq |F_{1\ldots\mu}|\,.$$

If this holds, then we have

$$\int_X |G| \leq \int_X |F| . \tag{1.5.2}$$

1.6 More spaces of differential forms

Let F be a differential form defined on a domain $\Omega \subset \mathbb{R}^\nu$. If F is a complex-valued function then

$$||F||(z) := |F(z)| \qquad \text{for all } z \in \Omega.$$

If $(x_1, \ldots, x_\nu)$ are canonical coordinates in $\mathbb{R}^\nu$ and

$$F = \sum_{\mathbf{j}} F_{\mathbf{j}} dx^{\mathbf{j}}$$

then

$$||F||(z) := \left(\sum_{\mathbf{j}} |F_{\mathbf{j}}(z)|^2 \right)^{\frac{1}{2}} .$$

Thus, the (Riemannian) norm of the differential form F at a point z is determined.

Given an **arbitrary set** Ω in $\mathbb{C}^m$ and a differential form F on it, we introduce the following natural definitions (see [HL2]):

$$||F||_0 := sup\{||F||(z) \,|z \in \Omega\}, \tag{1.6.1}$$

and for $0 < \alpha \leq 1$

$$||F||_\alpha := ||F||_0 + sup \left\{ \frac{\left(\sum_{\mathbf{j}} |F_{\mathbf{j}}(z) - F_{\mathbf{j}}(\zeta)|^2 \right)^{\frac{1}{2}}}{|z-\zeta|^\alpha} \;\middle|\; z \neq \zeta \right\}. \tag{1.6.2}$$

F is called α-Hölder continuous on Ω if

$$||F||_\alpha < \infty \tag{1.6.3}$$

for all compact subsets of Ω. Synonym: differential form of class $C^{0,\,\alpha}$. C^0 will stand for "continuous" differential forms. Now let Ω be a **domain** in $\mathbb{C}^m$. Given $k \in \mathbb{N}$, F is said to be a form of class C^k on $\overline{\Omega}$ if, for any α, F_α has all partial derivatives of orders up to k in Ω which extend continuously onto $\overline{\Omega}$. F is said to be of class C^∞ on $\overline{\Omega}$ if it is a form of class C^k on $\overline{\Omega}$ for any $k \in \mathbb{N} \cup \{0\}$. $C^{k,\,\alpha}(\Omega) \cap C^{k,\,\alpha}\left(\overline{\Omega}\right)$ stands for the subset of C^k on $\overline{\Omega}$ consisting of the differential forms on Ω being α-Hölder-continuous on $\overline{\Omega}$.

Chapter 2

Differential forms with coefficients in 2×2-matrices

2.1 Classes $\overline{\mathcal{G}}_p(\Omega)$, $\mathcal{G}_p(\Omega)$

Let Ω be a domain in $\mathbb{C}^m$ with the canonical complex coordinates $z = (z_1, \ldots, z_m)$. Given $k \in \{0, 1, \ldots, m\}$, $p \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, denote by $\overline{\mathcal{G}}_p^k(\Omega)$ the set of all $(0, k)$-forms on Ω of class C^p, and by $\mathcal{G}_p^s(\Omega)$ the set of all $(s, 0)$-forms of the same class; and set $\overline{\mathcal{G}}_p(\Omega) := \bigcup_{k=0}^{m} \overline{\mathcal{G}}_p^k(\Omega)$, $\mathcal{G}_p(\Omega) := \bigcup_{k=0}^{m} \mathcal{G}_p^k(\Omega)$. Natural operations of addition and of multiplication by complex scalars turn each of them into a complex linear space. Moreover, we shall consider $\overline{\mathcal{G}}_p(\Omega)$ as an algebra with respect to the exterior multiplication "$\wedge$"; thus, $\overline{\mathcal{G}}_p(\Omega)$ is a complex algebra which is associative, distributive, non-commutative, with zero-divisors and with identity. The same is true for $\mathcal{G}_p(\Omega)$.

2.2 Matrix-valued differential forms

Throughout the book, we shall deal with matrices whose entries are from different algebras which require careful distinction between matrix multiplication of two matrices and that of their elements. We shall use "$\star$" just for the matrix multiplication, providing it sometimes with precise symbols (subindex, superindex, etc.) related to

the multiplication in the algebra of their entries.

The main object of this paper is the set of 2×2 matrices with entries from $\overline{\mathcal{G}}_p(\Omega)$. Occasionally we shall consider its *symmetric* image replacing $\overline{\mathcal{G}}$ by $\mathcal{G}$. We use the following notations:

$$\begin{aligned}\overline{\mathfrak{G}}_p^k(\Omega) &:= \begin{pmatrix} \overline{\mathcal{G}}_p^k(\Omega) & \overline{\mathcal{G}}_p^k(\Omega) \\ \overline{\mathcal{G}}_p^k(\Omega) & \overline{\mathcal{G}}_p^k(\Omega) \end{pmatrix} \\ &:= \left\{ \begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{pmatrix} \mid \left\{F^{ij}\right\} \subset \overline{\mathcal{G}}_p^k(\Omega) \right\}\end{aligned}$$

and

$$\begin{aligned}\overline{\mathfrak{G}}_p(\Omega) &:= \begin{pmatrix} \overline{\mathcal{G}}_p(\Omega) & \overline{\mathcal{G}}_p(\Omega) \\ \overline{\mathcal{G}}_p(\Omega) & \overline{\mathcal{G}}_p(\Omega) \end{pmatrix} \\ &:= \left\{ \begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{pmatrix} \mid \left\{F^{ij}\right\} \subset \overline{\mathcal{G}}_p(\Omega) \right\}.\end{aligned}$$

The same for $\mathfrak{G}_p^s(\Omega)$ and $\mathfrak{G}_p(\Omega)$. The structure of a complex linear space in $\overline{\mathcal{G}}_p(\Omega)$ (and in $\mathcal{G}_p(\Omega)$) is inherited by $\overline{\mathfrak{G}}_p(\Omega)$ (and by $\mathfrak{G}_p(\Omega)$): it is enough to add the elements and to multiply them by complex scalars in an entry-wise manner. We will use sometimes the abbreviation **m.v.d.f.** for "matrix-valued differential form(s)".

Given F, G from $\overline{\mathfrak{G}}_p(\Omega)$ (or from $\mathfrak{G}_p(\Omega)$), their "exterior product" $F \underset{\star}{\wedge} G$ is introduced as follows:

$$\begin{aligned}F \underset{\star}{\wedge} G &= \begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{pmatrix} \underset{\star}{\wedge} \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix} \\ &:= \begin{pmatrix} F^{11} \wedge G^{11} + F^{12} \wedge G^{21}; & F^{11} \wedge G^{12} + F^{12} \wedge G^{22} \\ F^{21} \wedge G^{11} + F^{22} \wedge G^{21}; & F^{21} \wedge G^{12} + F^{22} \wedge G^{22} \end{pmatrix}.\end{aligned}$$

This product remains to be associative and distributive (it is straightforward to check it up):

$$\begin{aligned}\left(F \underset{\star}{\wedge} G\right) \underset{\star}{\wedge} H &= F \underset{\star}{\wedge} \left(G \underset{\star}{\wedge} H\right); \\ (F+G) \underset{\star}{\wedge} H &= F \underset{\star}{\wedge} H + G \underset{\star}{\wedge} H, \\ H \underset{\star}{\wedge} (F+G) &= H \underset{\star}{\wedge} F + H \underset{\star}{\wedge} G.\end{aligned}$$

At the same time, the *anti-commutativity rule* is now of the form

$$F \underset{\star}{\wedge} G = (-1)^{kk'} \left(G^{tr} \underset{\star}{\wedge} F^{tr} \right)^{tr}, \tag{2.2.1}$$

where $F \in \overline{\mathfrak{G}}_p^k(\Omega)$, $G \in \overline{\mathfrak{G}}_p^{k'}(\Omega)$ (or $F \in \mathfrak{G}_p^k(\Omega)$, $G \in \mathfrak{G}_p^{k'}(\Omega)$) and "$tr$" stands for transposing of the matrix.

Thus we shall consider $\overline{\mathfrak{G}}_p(\Omega)$ as a complex algebra which is associative, distributive, non-commutative, with zero divisors and with identity. The same with $\mathfrak{G}_p(\Omega)$.

2.3 The hyperholomorphic Cauchy-Riemann operators on $\overline{\mathfrak{G}}_1$ and $\mathfrak{G}_1$

Abusing perhaps a little the notation, we shall use the symbol "$\circ$" to denote a (well-defined) composition of any pair of operators we shall be in need of.

The differential operators introduced in Subsection 1.3 as operators acting on differential forms, extend naturally onto $\overline{\mathfrak{G}}_1(\Omega)$ and $\mathfrak{G}_1(\Omega)$ by their entry-wise actions, for instance, given $F \in \overline{\mathfrak{G}}_1(\Omega)$ or $F \in \mathfrak{G}_1(\Omega)$, we have

$$\begin{aligned} d[F] &:= \begin{pmatrix} d\left[F^{11}\right]; & d\left[F^{12}\right] \\ d\left[F^{21}\right]; & d\left[F^{22}\right] \end{pmatrix}, \\ \frac{\partial F}{\partial \bar{z}_j} &:= \begin{pmatrix} \sum\limits_{\mathbf{k}} \frac{\partial F_{\mathbf{k}}^{11}}{\partial \bar{z}_j} d\bar{z}^{\mathbf{k}}; & \sum\limits_{\mathbf{k}} \frac{\partial F_{\mathbf{k}}^{12}}{\partial \bar{z}_j} d\bar{z}^{\mathbf{k}} \\ \sum\limits_{\mathbf{k}} \frac{\partial F_{\mathbf{k}}^{21}}{\partial \bar{z}_j} d\bar{z}^{\mathbf{k}}; & \sum\limits_{\mathbf{k}} \frac{\partial F_{\mathbf{k}}^{22}}{\partial \bar{z}_j} d\bar{z}^{\mathbf{k}} \end{pmatrix}. \end{aligned}$$

Now we need certain matrix operators composed from *scalar* operators of Subsection 1.3 and acting on matrix-valued differential forms. We put

$$\overline{\mathcal{D}} := \begin{pmatrix} \bar{\partial} & \bar{\partial}^* \\ \bar{\partial}^* & \bar{\partial} \end{pmatrix}, \qquad \overline{\mathcal{D}}^* := \begin{pmatrix} \bar{\partial}^* & \bar{\partial} \\ \bar{\partial} & \bar{\partial}^* \end{pmatrix}; \tag{2.3.1}$$

and similarly

$$\mathcal{D} := \begin{pmatrix} \partial & \partial^* \\ \partial^* & \partial \end{pmatrix}, \qquad \mathcal{D}^* := \begin{pmatrix} \partial^* & \partial \\ \partial & \partial^* \end{pmatrix}; \tag{2.3.2}$$

i.e., for $F \in \overline{\mathfrak{G}}_1(\Omega)$ we define $\overline{\mathcal{D}}[F]$ and $\overline{\mathcal{D}}^*[F]$ to be

$$\begin{aligned}\overline{\mathcal{D}}[F] &= \begin{pmatrix} \bar\partial \wedge F^{11} + \bar\partial^* \wedge F^{21}; & \bar\partial \wedge F^{12} + \bar\partial^* \wedge F^{22} \\ \bar\partial^* \wedge F^{11} + \bar\partial \wedge F^{21}; & \bar\partial^* \wedge F^{12} + \bar\partial \wedge F^{22} \end{pmatrix} = \\ &= \begin{pmatrix} \bar\partial \left[F^{11}\right] + \bar\partial^* \left[F^{21}\right]; & \bar\partial \left[F^{12}\right] + \bar\partial^* \left[F^{22}\right] \\ \bar\partial^* \left[F^{11}\right] + \bar\partial \left[F^{21}\right]; & \bar\partial^* \left[F^{12}\right] + \bar\partial \left[F^{22}\right] \end{pmatrix}, \end{aligned} \tag{2.3.3}$$

$$\begin{aligned}\overline{\mathcal{D}}^*[F] &= \begin{pmatrix} \bar\partial^* \wedge F^{11} + \bar\partial \wedge F^{21}; & \bar\partial^* \wedge F^{12} + \bar\partial \wedge F^{22} \\ \bar\partial \wedge F^{11} + \bar\partial^* \wedge F^{21}; & \bar\partial \wedge F^{12} + \bar\partial^* \wedge F^{22} \end{pmatrix} = \\ &= \begin{pmatrix} \bar\partial^* \left[F^{11}\right] + \bar\partial \left[F^{21}\right]; & \bar\partial^* \left[F^{12}\right] + \bar\partial \left[F^{22}\right] \\ \bar\partial \left[F^{11}\right] + \bar\partial^* \left[F^{21}\right]; & \bar\partial \left[F^{12}\right] + \bar\partial^* \left[F^{22}\right] \end{pmatrix}, \end{aligned} \tag{2.3.4}$$

analogously for $\mathcal{D}$ and $\mathcal{D}^*$.

Let I be the identity operator acting on some linear space of differential forms; then we shall denote by $\mathbb{E}_{2\times 2}$ and $\breve{\mathbb{E}}_{2\times 2}$, respectively, the operators of the (left-hand-side) multiplication by $E_{2\times 2}$ and $\breve{E}_{2\times 2}$ (see Subsection 1.1) on the corresponding linear space of **m.v.d.f.**, i.e.,

$$\mathbb{E}_{2\times 2} ; = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \breve{\mathbb{E}}_{2\times 2} := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Then the following operator equalities hold on $\overline{\mathfrak{G}}_2(\mathbb{C}^m)$ and $\mathfrak{G}_2(\mathbb{C}^m)$, respectively; they are of extreme importance for the whole theory:

$$\overline{\mathcal{D}} \circ \overline{\mathcal{D}}^* = \overline{\mathcal{D}}^* \circ \overline{\mathcal{D}} = \Delta_{\mathbb{C}^m} \mathbb{E}_{2\times 2}, \tag{2.3.5}$$

and similarly,

$$\mathcal{D} \circ \mathcal{D}^* = \mathcal{D}^* \circ \mathcal{D} = \Delta_{\mathbb{C}^m} \mathbb{E}_{2\times 2}. \tag{2.3.6}$$

Recalling the observation in Subsection 1.3, we can interpret the matrix $\overline{\mathcal{D}}[F]$ as a result of the "matrix wedge multiplication" of F by $\overline{\mathcal{D}}$ on the left-hand-side:

$$\overline{\mathcal{D}} \underset{\star}{\wedge} F := \begin{pmatrix} \bar\partial & \bar\partial^* \\ \bar\partial^* & \bar\partial \end{pmatrix} \underset{\star}{\wedge} \begin{pmatrix} F^{11} & F^{12} \\ F^{21} & F^{22} \end{pmatrix}.$$

Now we introduce *the right-hand side* operator $\overline{\mathcal{D}}_r$ by the rule

$$\overline{\mathcal{D}}_r\,[F] \;:=\; \left(\begin{array}{ll} \bar\partial_r\left[F^{11}\right] + \bar\partial_r^*\left[F^{12}\right]; & \bar\partial_r^*\left[F^{11}\right] + \bar\partial_r\left[F^{12}\right] \\ \bar\partial_r\left[F^{21}\right] + \bar\partial_r^*\left[F^{22}\right]; & \bar\partial_r^*\left[F^{21}\right] + \bar\partial_r\left[F^{22}\right] \end{array} \right), \quad (2.3.7)$$

which may differ greatly from $\overline{\mathcal{D}} \underset{\star}{\wedge} F \;=\; \overline{\mathcal{D}}\,[F]$: the latter is an **m.v.d.f.** while the former is a family of operators acting on **m.v.d.f.**, which depends of $z \in \Omega$. Analogous definitions and conclusions are true for the *right-hand-side* operators $\overline{\mathcal{D}}_r^*$, $\mathcal{D}_r$, $\mathcal{D}_r^*$.

The above operators, $\overline{\mathcal{D}}$, $\overline{\mathcal{D}}^*$, $\mathcal{D}$, and $\mathcal{D}^*$, as well as their right-hand-side counterparts, are called the hyperholomorphic Cauchy-Riemann operators, although the right-hand-side case has its peculiarities which will be explained later.

The equality (2.3.5) may be seen as a factorization of the matrix Laplace operator,

$$\Delta_{\mathbb{C}^m}\,\mathbb{E}_{2\times 2} \;=\; \left(\begin{array}{ll} \Delta_{\mathbb{C}^m} & 0 \\ 0 & \Delta_{\mathbb{C}^m} \end{array} \right).$$

There are other ways of factorizing the matrix Laplace operator. Indeed, the operators $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}^*$ are not independent:

$$\overline{\mathcal{D}}^* \;=\; \overline{\mathcal{D}} \circ \breve{\mathbb{E}}_{2\times 2} \;=\; \breve{\mathbb{E}}_{2\times 2} \circ \overline{\mathcal{D}}, \quad (2.3.8)$$

or equivalently

$$\overline{\mathcal{D}} \;=\; \overline{\mathcal{D}}^* \circ \breve{\mathbb{E}}_{2\times 2} \;=\; \breve{\mathbb{E}}_{2\times 2} \circ \overline{\mathcal{D}}^*. \quad (2.3.9)$$

This leads to factorizations of another matrix Laplace operator, $\Delta\breve{\mathbb{E}}_{2\times 2} \;=\; \left(\begin{array}{ll} 0 & \Delta_{\mathbb{C}^m} \\ \Delta_{\mathbb{C}^m} & 0 \end{array} \right)$:

$$\overline{\mathcal{D}} \circ \overline{\mathcal{D}} \;=\; \Delta_{\mathbb{C}^m} \cdot \breve{\mathbb{E}}_{2\times 2};$$

$$\overline{\mathcal{D}}^* \circ \overline{\mathcal{D}}^* \;=\; \Delta_{\mathbb{C}^m} \cdot \breve{\mathbb{E}}_{2\times 2}.$$

The same type of relations hold for $\mathcal{D}$ and $\mathcal{D}^*$; we chose the factorization (2.3.5) just to fix one of them.

2.4 Formula for $d\left(F \underset{\star}{\wedge} G\right)$

Let $F \in \overline{\mathfrak{G}}_1^k$, $G \in \overline{\mathfrak{G}}_1^s$, then for any pair of their entries we have that

$$d\left(F^{\alpha\beta} \wedge G^{\gamma\delta}\right) = dF^{\alpha\beta} \wedge G_{\gamma\delta} + (-1)^k F^{\alpha\beta} \wedge dG^{\gamma\delta}.$$

It is straightforward now to verify that the same is true for matrices:

$$d\left(F \underset{\star}{\wedge} G\right) = dF \underset{\star}{\wedge} G + (-1)^k F \underset{\star}{\wedge} dG.$$

The same formula is valid for $F \in \mathfrak{G}_1^k$, $G \in \mathfrak{G}_1^s$.

2.5 Differential matrix forms of the unit normal

The following operators acting on **m.v.d.f.** are of special importance. Let $\zeta = (\zeta_1, \ \ldots, \ \zeta_m)$ and $z = (z_1, \ \ldots, \ z_m)$ be canonical coordinates in spaces $\mathbb{C}_\zeta^m$ and $\mathbb{C}_z^m$ respectively. Then the following objects are defined (for $(\zeta, \ z) \in \mathbb{C}_\zeta^m \times \mathbb{C}_z^m$):

$$\begin{aligned}
\bar{\sigma} &:= \bar{\sigma}_{\zeta,\, z} = \sum_{j=1}^{m} c_j d\bar{\zeta}_{[j]} \wedge d\zeta \wedge d\bar{z}^j, \\
\bar{\sigma}^* &:= \bar{\sigma}^*_{\zeta,\, z} = (-1)^m \sum_{j=1}^{m} c_j d\bar{\zeta} \wedge d\zeta_{[j]} \wedge \widehat{d\bar{z}^j},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\sigma &:= \sigma_{\zeta,\, z} = (-1)^m \sum_{j=1}^{m} c_j d\bar{\zeta} \wedge d\zeta_{[j]} \wedge dz^j, \\
\sigma^* &:= \sigma^*_{\zeta,\, z} = \sum_{j=1}^{m} c_j d\bar{\zeta}_{[j]} \wedge d\zeta \wedge \widehat{dz^j},
\end{aligned}$$

where

$$c_j := \frac{(-1)^{\frac{m(m-1)}{2}}}{(2i)^m} (-1)^{j-1}.$$

They will serve as entries of the following matrices:

$$\begin{aligned}\overline{\boldsymbol{\sigma}} &:= \overline{\boldsymbol{\sigma}}_{\zeta,\, z} = \begin{pmatrix} \bar{\sigma} & \bar{\sigma}^* \\ \bar{\sigma}^* & \bar{\sigma} \end{pmatrix}, \\ \overline{\boldsymbol{\sigma}}^* &:= \overline{\boldsymbol{\sigma}}^*_{\zeta,\, z} = \begin{pmatrix} \bar{\sigma}^* & \bar{\sigma} \\ \bar{\sigma} & \bar{\sigma}^* \end{pmatrix},\end{aligned}$$

and similarly:

$$\begin{aligned}\boldsymbol{\sigma} &:= \boldsymbol{\sigma}_{\zeta,\, z} = \begin{pmatrix} \sigma & \sigma^* \\ \sigma^* & \sigma \end{pmatrix}, \\ \boldsymbol{\sigma}^* &:= \boldsymbol{\sigma}^*_{\zeta,\, z} = \begin{pmatrix} \sigma^* & \sigma \\ \sigma & \sigma^* \end{pmatrix}.\end{aligned}$$

The structure of all these matrices shows that there is a relation like (2.3.8) between $\bar{\boldsymbol{\sigma}}^*$ and $\bar{\boldsymbol{\sigma}}$ as well as between $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$. By definition, all symbols $d\zeta^j$, $d\bar{\zeta}^j$ commute with all symbols $d\bar{z}^j$, $\widehat{d\bar{z}^j}$, dz^j, $\widehat{dz^j}$. Recalling the definition of the contraction operators $\widehat{d\bar{z}_j}$ and $\widehat{dz_j}$, we see that $\overline{\boldsymbol{\sigma}}$, $\overline{\boldsymbol{\sigma}}^*$ and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^*$ should be seen as operators on some spaces of **m.v.d.f.** F with entries $F^{ij}(\zeta,\, z) = \sum\limits_{\alpha,\, \beta} F^{ij}_{\alpha,\, \beta}(\zeta,\, z) \wedge dz^\alpha \wedge d\bar{z}^\beta$ ($\zeta \in \Lambda_1$, $z \in \Lambda_2$) and each of $F^{ij}_{\alpha,\, \beta} = F^{ij}_{\alpha,\, \beta}(\cdot,\, z)$ is a family of **d.f.** on Λ_1 which depends on $z \in \Lambda_2$, i.e.,

$$\overline{\boldsymbol{\sigma}}[F] := \overline{\boldsymbol{\sigma}} \underset{\star}{\wedge} F := \begin{pmatrix} \bar{\sigma} \wedge F^{11} + \bar{\sigma}^* \wedge F^{21}; & \bar{\sigma} \wedge F^{12} + \bar{\sigma}^* \wedge F^{22} \\ \bar{\sigma}^* \wedge F^{11} + \bar{\sigma} \wedge F^{21}; & \bar{\sigma}^* \wedge F^{12} + \bar{\sigma} \wedge F^{22} \end{pmatrix}$$

where

$$\begin{aligned}&\bar{\sigma}\left[F^{ij}\right] := \bar{\sigma} \wedge F^{ij} := \\ &\quad := \sum_{\alpha,\, \beta} \sum_{j=1}^{m} c_j d\bar{\zeta}_{[j]} \wedge d\zeta \wedge F^{ij}_{\alpha,\, \beta} \wedge d\bar{z}^j \wedge dz^\alpha \wedge d\bar{z}^\beta,\end{aligned}$$

$$\begin{aligned}&\bar{\sigma}^*\left[F^{ij}\right] := \bar{\sigma}^* \wedge F^{ij} := \\ &\quad := \sum_{\alpha,\, \beta} \sum_{j=1}^{m} c_j d\bar{\zeta} \wedge d\zeta_{[j]} \wedge F^{ij}_{\alpha,\, \beta} \wedge \widehat{d\bar{z}^j}\left[dz^\alpha \wedge d\bar{z}^\beta\right].\end{aligned}$$

This means, in particular, that here we identify the differential forms $\bar{\sigma}$ and σ with the operators of (left) multiplication by them.

Consider now the relations between the above-introduced matrices $\overline{\boldsymbol{\sigma}}$, $\overline{\boldsymbol{\sigma}}^*$, and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^*$, and the normal vector to a surface in $\mathbb{C}^m$. Let Γ be a real $(2m-1)$-surface in $\mathbb{C}^m$ of class C^1. Denote by $\mathbf{n}_\zeta = \big(n_{1\zeta}, \ldots, n_{m\zeta}\big)$ the outward pointing normal unit vector to Γ at $\zeta \in \Gamma$ and let dS be a surface differential form on Γ. Consider now on the surface Γ

$$
\begin{aligned}
n_j dS \;=\; & \Big\{(-1)^{(2j-1)-1}\,\frac{1}{2}\left(d\bar{\zeta}_1 + d\zeta_1\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_1 - d\zeta_1\right) \wedge \ldots \\
& \ldots \wedge \frac{1}{2}\left(d\bar{\zeta}_{j-1} + d\zeta_{j-1}\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_{j-1} - d\zeta_{j-1}\right) \wedge \\
& \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_j - d\zeta_j\right) \wedge \\
& \wedge \frac{1}{2}\left(d\bar{\zeta}_{j+1} + d\zeta_{j+1}\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_{j+1} - d\zeta_{j+1}\right) \wedge \ldots \\
& \ldots \wedge \frac{1}{2}\left(d\bar{\zeta}_m + d\zeta_m\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_m - d\zeta_m\right) + \\
& + (-1)^{(2j)-1}\, i\frac{1}{2}\left(d\bar{\zeta}_1 + d\zeta_1\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_1 - d\zeta_1\right) \wedge \ldots \\
& \ldots \wedge \frac{1}{2}\left(d\bar{\zeta}_{j-1} + d\zeta_{j-1}\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_{j-1} - d\zeta_{j-1}\right) \wedge \\
& \wedge \frac{1}{2}\left(d\bar{\zeta}_j + d\zeta_j\right) \wedge \\
& \wedge \frac{1}{2}\left(d\bar{\zeta}_{j+1} + d\zeta_{j+1}\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_{j+1} - d\zeta_{j+1}\right) \wedge \ldots \\
& \ldots \wedge \frac{1}{2}\left(d\bar{\zeta}_m + d\zeta_m\right) \wedge \left(-\frac{1}{2i}\right)\left(d\bar{\zeta}_m - d\zeta_m\right)\Big\}\,|\Gamma = \\
= \; & \Big\{\frac{(-1)^m}{2^{2m-1} i m}\left(d\bar{\zeta}_j - d\zeta_j\right) \wedge \bigwedge_{\substack{k=1 \\ k\neq j}}^{m} (-2)\, d\bar{\zeta}_k \wedge d\zeta_k - \\
& - \frac{(-1)^m}{2^{2m-1} i m}\left(d\bar{\zeta}_j + d\zeta_j\right) \wedge \bigwedge_{\substack{k=1 \\ k\neq j}}^{m} (-2)\, d\bar{\zeta}_k \wedge d\zeta_k \Big\}\,|\Gamma =
\end{aligned}
$$

$$= \quad 2\frac{1}{(2i)^m}\left\{ d\zeta_j \wedge \bigwedge_{\substack{k=1\\k\neq j}}^{m} d\bar{\zeta}_k \wedge d\zeta_k \right\} |\Gamma =$$

$$= \quad 2\frac{(-1)^{\frac{(m-2)(m-1)}{2}}}{(2i)^m}\left\{ d\zeta_j \wedge d\bar{\zeta}_{[j]} \wedge d\zeta_{[j]} \right\} |\Gamma =$$

$$= \quad 2\frac{(-1)^{\frac{m(m-1)}{2}}}{(2i)^m}(-1)^{j-1}\left\{ d\bar{\zeta}_{[j]} \wedge d\zeta \right\} |\Gamma.$$

Therefore

$$n_j dS = 2\frac{(-1)^{\frac{m(m-1)}{2}}}{(2i)^m}(-1)^{j-1}\left\{ d\bar{\zeta}_{[j]} \wedge d\zeta \right\} |\Gamma.$$

Analogously, we have

$$\bar{n}_j dS = 2\,(-1)^m \frac{(-1)^{\frac{m(m-1)}{2}}}{(2i)^m}(-1)^{j-1}\left\{ d\bar{\zeta} \wedge d\zeta_{[j]} \right\} |\Gamma,$$

and hence

$$\begin{aligned}
\overline{\boldsymbol{\sigma}}|\Gamma &= \frac{1}{2}\sum_{j=1}^{m}\begin{pmatrix} n_j d\bar{z}^j; & \bar{n}_j \widehat{d\bar{z}^j} \\ \bar{n}_j \widehat{d\bar{z}^j}; & n_j d\bar{z}^j \end{pmatrix} dS_\zeta = \\
&= \frac{1}{2}\sum_{j=1}^{m}\left(n_j d\bar{z}^j E_{2\times 2} + \bar{n}_j \widehat{d\bar{z}^j}\breve{E}_{2\times 2}\right) dS_\zeta, \qquad (2.5.1)\\
\overline{\boldsymbol{\sigma}}^*|\Gamma &= \frac{1}{2}\sum_{j=1}^{m}\begin{pmatrix} \bar{n}_j \widehat{d\bar{z}^j}; & n_j d\bar{z}^j \\ n_j d\bar{z}^j; & \bar{n}_j \widehat{d\bar{z}^j} \end{pmatrix} dS_\zeta, \qquad (2.5.2)
\end{aligned}$$

and symmetrically

$$\begin{aligned}
\boldsymbol{\sigma}|\Gamma &= \frac{1}{2}\sum_{j=1}^{m}\begin{pmatrix} \bar{n}_j d\bar{z}^j; & n_j \widehat{d\bar{z}^j} \\ n_j \widehat{d\bar{z}^j}; & \bar{n}_j d\bar{z}^j \end{pmatrix} dS_\zeta, \qquad (2.5.3)\\
\boldsymbol{\sigma}^*|\Gamma &= \frac{1}{2}\sum_{j=1}^{m}\begin{pmatrix} n_j \widehat{d\bar{z}^j}; & \bar{n}_j d\bar{z}^j \\ \bar{n}_j d\bar{z}^j; & n_j \widehat{d\bar{z}^j} \end{pmatrix} dS_\zeta. \qquad (2.5.4)
\end{aligned}$$

Thus these matrices will serve for integrating **m.v.d.f.** of two variables, ζ and z, with respect to ζ over surfaces in $\mathbb{C}_{\zeta}^{m}$.

2.6 Formula for $d_{\zeta}\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}} \underset{\star}{\wedge} G\right)$

Let F and G be two elements from $\overline{\mathfrak{G}}_1(\Omega)$, consider

$$d_{\zeta}\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}} \underset{\star}{\wedge} G\right) = d_{\zeta}\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}\,[G]\right).$$

For any $F(\zeta,\ d\bar{z})$, $G(\zeta,\ d\bar{z})$ we have

$$F(\zeta,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\zeta,\ z} \underset{\star}{\wedge} G(\zeta,\ d\bar{z}) = \begin{pmatrix} A^{11}_{\zeta,\ z} & A^{12}_{\zeta,\ z} \\ A^{21}_{\zeta,\ z} & A^{22}_{\zeta,\ z} \end{pmatrix}$$

with

$$\begin{aligned} A^{11}_{\zeta,\ z} \;:=\; & F^{11}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}_{\zeta,\ z} \wedge G^{11}(\zeta,\ d\bar{z}) + \\ & + F^{11}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}^{*}_{\zeta,\ z} \wedge G^{21}(\zeta,\ d\bar{z}) + \\ & + F^{12}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}^{*}_{\zeta,\ z} \wedge G^{11}(\zeta,\ d\bar{z}) + \\ & + F^{12}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}_{\zeta,\ z} \wedge G^{21}(\zeta,\ d\bar{z}); \end{aligned}$$

$$\begin{aligned} A^{12}_{\zeta,\ z} \;:=\; & F^{11}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}_{\zeta,\ z} \wedge G^{12}(\zeta,\ d\bar{z}) + \\ & + F^{11}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}^{*}_{\zeta,\ z} \wedge G^{22}(\zeta,\ d\bar{z}) + \\ & + F^{12}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}^{*}_{\zeta,\ z} \wedge G^{12}(\zeta,\ d\bar{z}) + \\ & + F^{12}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}_{\zeta,\ z} \wedge G^{22}(\zeta,\ d\bar{z}); \end{aligned}$$

$$\begin{aligned} A^{21}_{\zeta,\ z} \;:=\; & F^{21}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}_{\zeta,\ z} \wedge G^{11}(\zeta,\ d\bar{z}) + \\ & + F^{21}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}^{*}_{\zeta,\ z} \wedge G^{21}(\zeta,\ d\bar{z}) + \\ & + F^{22}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}^{*}_{\zeta,\ z} \wedge G^{11}(\zeta,\ d\bar{z}) + \\ & + F^{22}(\zeta,\ d\bar{z}) \wedge \bar{\sigma}_{\zeta,\ z} \wedge G^{21}(\zeta,\ d\bar{z}); \end{aligned}$$

$$d_\zeta\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}} \underset{\star}{\wedge} G\right)$$

$$\begin{aligned} A^{22}_{\zeta,\,z} \;:=\; & F^{21}(\zeta,\,d\bar z)\wedge \bar\sigma_{\zeta,\,z}\wedge G^{12}(\zeta,\,d\bar z)+\\ & +F^{21}(\zeta,\,d\bar z)\wedge \bar\sigma^{*}_{\zeta,\,z}\wedge G^{22}(\zeta,\,d\bar z)+\\ & +F^{22}(\zeta,\,d\bar z)\wedge \bar\sigma^{*}_{\zeta,\,z}\wedge G^{12}(\zeta,\,d\bar z)+\\ & +F^{22}(\zeta,\,d\bar z)\wedge \bar\sigma_{\zeta,\,z}\wedge G^{22}(\zeta,\,d\bar z)\,. \end{aligned}$$

Because of linearity of d it is enough to consider

$$F(\zeta,\,d\bar z)=\left(F^{\alpha\beta}(\zeta,d\bar z)\right)^{1\le\beta\le 2}_{1\le\alpha\le 2},$$

$$G(\zeta,\,d\bar z)=\left(G^{\gamma\delta}(\zeta,\,d\bar z)\right)^{1\le\delta\le 2}_{1\le\gamma\le 2},$$

with entries of the form

$$F^{\alpha\beta}(\zeta,\,d\bar z):=\varphi_{\alpha\beta}(\zeta)\,d\bar z^{\mathbf{j}^{\alpha\beta}}$$

and

$$G^{\gamma\delta}(\zeta,\,d\bar z):=\psi_{\gamma\delta}(\zeta)\,d\bar z^{\mathbf{q}^{\gamma\delta}},$$

where $\mathbf{j}^{\alpha\beta}=\left(j^{\alpha\beta}_1,\,\ldots,\,j^{\alpha\beta}_{k_{\alpha\beta}}\right)$, $\mathbf{q}^{\gamma\delta}=\left(q^{\gamma\delta}_1,\,\ldots,\,q^{\gamma\delta}_{p_{\gamma\delta}}\right)$ and $\varphi_{\alpha\beta}(\zeta)$, $\psi_{\gamma\delta}(\zeta)$ are functions of class C^1.

Take $F^{11}(\zeta,\,d\bar z)=\varphi_{11}(\zeta)\,d\bar z^{\mathbf{j}^{11}}$ and $G^{11}(\zeta,\,d\bar z)=\psi_{11}(\zeta)\,d\bar z^{\mathbf{q}^{11}}$,

$$\begin{aligned} & d_\zeta\left(F^{11}\wedge\bar\sigma^{*}\wedge G^{11}\right)=\\ &= d_\zeta\left(\varphi_{11}d\bar z^{\mathbf{j}^{11}}\wedge\left((-1)^m\sum_{s=1}^{m}c_s d\bar\zeta\wedge d\zeta_{[s]}\wedge\widehat{d\bar z^{s}}\right)\wedge\right.\\ &\qquad\left.\wedge\psi_{11}d\bar z^{\mathbf{q}^{11}}\right)=\\ &= (-1)^m\sum_{s=1}^{m}c_s d_\zeta\left(\varphi_{11}\psi_{11}d\bar\zeta\wedge d\zeta_{[s]}\right)\wedge\left(d\bar z^{\mathbf{j}^{11}}\wedge\widehat{d\bar z^{s}}\wedge d\bar z^{\mathbf{q^{11}}}\right)=\\ &= (-1)^m\sum_{s=1}^{m}c_s\frac{\partial(\varphi_{11}\psi_{11})}{\partial\zeta_s}(-1)^{m+s-1}\,d\bar\zeta\wedge d\zeta\wedge\left(d\bar z^{\mathbf{j}^{11}}\wedge\widehat{d\bar z^{s}}\wedge\right.\\ &\qquad\left.\wedge\,d\bar z^{\mathbf{q^{11}}}\right)=\\ &= (-1)^m\sum_{s=1}^{m}c_s\left(\frac{\partial\varphi_{11}}{\partial\zeta_s}\psi_{11}+\varphi_{11}\frac{\partial\psi_{11}}{\partial\zeta_s}\right)(-1)^{m+s-1}\,d\bar\zeta\wedge d\zeta\wedge \end{aligned}$$

$$\wedge \left(d\bar{z}^{\mathbf{j}^{11}} \wedge \widehat{d\bar{z}^{s}} \wedge d\bar{z}^{\mathbf{q}^{\mathbf{11}}} \right).$$

Completely analogously we have

$$\begin{aligned} & d_{\zeta} \left(F^{\alpha\beta} \wedge \bar{\sigma}^{*} \wedge G^{\gamma\delta} \right) = \\ & = \quad (-1)^{m} \sum_{s=1}^{m} c_{s} \left(\frac{\partial \varphi_{\alpha\beta}}{\partial \zeta_{s}} \psi_{\gamma\delta} + \varphi_{\alpha\beta} \frac{\partial \psi_{\gamma\delta}}{\partial \zeta_{s}} \right) (-1)^{m+s-1} \, d\bar{\zeta} \wedge d\zeta \wedge \\ & \qquad \wedge \left(d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge \widehat{d\bar{z}^{s}} \wedge d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right), \end{aligned}$$

for all the other possible combinations of indices.

Consider now

$$\begin{aligned} & d_{\zeta} \left(F^{\alpha\beta} \wedge \bar{\sigma} \wedge G^{\gamma\delta} \right) = \\ & = \quad d_{\zeta} \left(\varphi_{\alpha\beta} d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge \left(\sum_{s=1}^{m} c_{s} d\bar{\zeta}_{[s]} \wedge d\zeta \wedge d\bar{z}^{s} \right) \wedge \psi_{\alpha\beta} d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right) = \\ & = \quad \sum_{s=1}^{m} c_{s} d_{\zeta} \left(\varphi_{\alpha\beta} \psi_{\gamma\delta} d\bar{\zeta}_{[s]} \wedge d\zeta \right) \wedge \left(d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge d\bar{z}^{s} \wedge d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right) = \\ & = \quad \sum_{s=1}^{m} c_{s} \frac{\partial \left(\varphi_{\alpha\beta} \psi_{\gamma\delta} \right)}{\partial \bar{\zeta}_{s}} (-1)^{s-1} \, d\bar{\zeta} \wedge d\zeta \wedge \left(d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge d\bar{z}^{s} \wedge d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right) = \\ & = \quad \sum_{s=1}^{m} c_{s} \left(\frac{\partial \varphi_{\alpha\beta}}{\partial \bar{\zeta}_{s}} \psi_{\gamma\delta} + \varphi_{\alpha\beta} \frac{\partial \psi_{\gamma\delta}}{\partial \bar{\zeta}_{s}} \right) (-1)^{s-1} \, d\bar{\zeta} \wedge d\zeta \wedge \left(d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge \right. \\ & \qquad \left. \wedge d\bar{z}^{s} \wedge d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right). \end{aligned}$$

Completely analogously for $F,\ G \in \mathfrak{G}_{1}(\Omega)$ we have

$$\begin{aligned} & d_{\zeta} \left(F^{\alpha\beta} \wedge \sigma \wedge G^{\gamma\delta} \right) = \\ & = \quad (-1)^{m} \sum_{s=1}^{m} c_{s} \left(\frac{\partial \varphi_{\alpha\beta}}{\partial \zeta_{s}} \psi_{\gamma\delta} + \varphi_{\alpha\beta} \frac{\partial \psi_{\gamma\delta}}{\partial \zeta_{s}} \right) (-1)^{m+s-1} \, d\bar{\zeta} \wedge d\zeta \wedge \\ & \qquad \wedge \left(d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge d\bar{z}^{s} \wedge d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right), \\ & d_{\zeta} \left(F^{\alpha\beta} \wedge \sigma^{*} \wedge G^{\gamma\delta} \right) = \end{aligned}$$

$$= \sum_{s=1}^{m} c_s \left(\frac{\partial \varphi_{\alpha\beta}}{\partial \bar{\zeta}_s} \psi_{\gamma\delta} + \varphi_{\alpha\beta} \frac{\partial \psi_{\gamma\delta}}{\partial \bar{\zeta}_s} \right) (-1)^{s-1} \, d\bar{\zeta} \wedge d\zeta \wedge$$
$$\wedge \left(d\bar{z}^{\mathbf{j}^{\alpha\beta}} \wedge \widehat{d\bar{z}^s} \wedge d\bar{z}^{\mathbf{q}^{\gamma\delta}} \right).$$

Hence we have, respectively, the relations

$d_\zeta \left(F^{\alpha\beta} \wedge \bar{\sigma} \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$

$$= -d_{r\zeta} \left(F^{\alpha\beta} \wedge \bar{\sigma} \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$$
$$= \left(\bar{\partial}_r \left[F^{\alpha\beta} \right] (\zeta, \, d\bar{z}) \wedge G^{\gamma\delta} \, (\zeta, \, d\bar{z}) \right.$$
$$\left. + F^{\alpha\beta} \, (\zeta, \, d\bar{z}) \wedge \bar{\partial} \left[G^{\gamma\delta} \right] (\zeta, \, d\bar{z}) \right) dV_\zeta,$$

$d_\zeta \left(F^{\alpha\beta} \wedge \bar{\sigma}^* \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$

$$= -d_{r\,\zeta} \left(F^{\alpha\beta} \wedge \bar{\sigma}^* \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$$
$$= \left(\bar{\partial}_r^* \left[F^{\alpha\beta} \right] (\zeta, \, d\bar{z}) \wedge G^{\gamma\delta} \, (\zeta, \, d\bar{z}) + \right.$$
$$\left. + F^{\alpha\beta} \, (\zeta, \, d\bar{z}) \wedge \bar{\partial}^* \left[G^{\gamma\delta} \right] (\zeta, \, d\bar{z}) \right) dV_\zeta,$$

$d_\zeta \left(F^{\alpha\beta} \wedge \sigma \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$

$$= -d_{r\,\zeta} \left(F^{\alpha\beta} \wedge \sigma \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$$
$$= \left(\partial_r \left[F^{\alpha\beta} \right] (\zeta, \, d\bar{z}) \wedge G^{\gamma\delta} \, (\zeta, \, d\bar{z}) + \right.$$
$$\left. + F^{\alpha\beta} \, (\zeta, \, d\bar{z}) \wedge \partial \left[G^{\gamma\delta} \right] (\zeta, \, d\bar{z}) \right) dV_\zeta,$$

$d_\zeta \left(F^{\alpha\beta} \wedge \sigma^* \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$

$$= -d_{r\,\zeta} \left(F^{\alpha\beta} \wedge \sigma^* \wedge G^{\gamma\delta} \right) (\zeta, \, d\bar{z}) =$$
$$= \left(\partial_r^* \left[F^{\alpha\beta} \right] (\zeta, \, d\bar{z}) \wedge G^{\gamma\delta} \, (\zeta, \, d\bar{z}) + \right.$$
$$\left. + F^{\alpha\beta} \, (\zeta, \, d\bar{z}) \wedge \partial^* \left[G^{\gamma\delta} \right] (\zeta, \, d\bar{z}) \right) dV_\zeta.$$

2.7 Exterior differentiation and the hyperholomorphic Cauchy-Riemann operators

Theorem *Let F and G be arbitrary* **m.v.d.f.** *from $\overline{\mathfrak{G}}_1(\Omega)$ and F', G' be arbitrary* **m.v.d.f.** *from $\mathfrak{G}_1(\Omega)$. The following equalities hold:*

$$\begin{aligned} d_\zeta\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}} \underset{\star}{\wedge} G\right) &= d_\zeta\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}\,[G]\right) = -d_{r\,\zeta}\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}} \underset{\star}{\wedge} G\right) = \\ &= \left(\overline{\mathcal{D}}_r\,[F] \underset{\star}{\wedge} G + F \underset{\star}{\wedge} \overline{\mathcal{D}}\,[G]\right) dV_\zeta, \end{aligned} \tag{2.7.1}$$

$$\begin{aligned} d_\zeta\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^* \underset{\star}{\wedge} G\right) &= -d_{r\,\zeta}\left(F \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^* \underset{\star}{\wedge} G\right) = \\ &= \left(\overline{\mathcal{D}}_r^*\,[F] \underset{\star}{\wedge} G + F \underset{\star}{\wedge} \overline{\mathcal{D}}^*\,[G]\right) dV_\zeta, \end{aligned} \tag{2.7.2}$$

$$\begin{aligned} d_\zeta\left(F' \underset{\star}{\wedge} \boldsymbol{\sigma} \underset{\star}{\wedge} G'\right) &= -d_{r\,\zeta}\left(F' \underset{\star}{\wedge} \boldsymbol{\sigma} \underset{\star}{\wedge} G'\right) = \\ &= \left(\mathcal{D}_r\left[F'\right] \underset{\star}{\wedge} G' + F' \underset{\star}{\wedge} \mathcal{D}\left[G'\right]\right) dV_\zeta, \end{aligned} \tag{2.7.3}$$

$$\begin{aligned} d_\zeta\left(F' \underset{\star}{\wedge} \boldsymbol{\sigma}^* \underset{\star}{\wedge} G'\right) &= -d_{r\,\zeta}\left(F' \underset{\star}{\wedge} \boldsymbol{\sigma}^* \underset{\star}{\wedge} G'\right) = \\ &= \left(\mathcal{D}_r^*\left[F'\right] \underset{\star}{\wedge} G' + F' \underset{\star}{\wedge} \mathcal{D}^*\left[G'\right]\right) dV_\zeta. \end{aligned} \tag{2.7.4}$$

Proof. Compare the left-hand sides of each equality with their right-hand sides, using formulas in the end of the last section. ■

2.8 Stokes formula compatible with the hyperholo morphic Cauchy-Riemann operators

Theorem *Let Ω^+ be a bounded domain in $\mathbb{C}^m$ with the topological boundary Γ, which is a piecewise smooth surface. Let F and G be two arbitrary*

m.v.d.f. *from* $\overline{\mathfrak{G}}_1(\Omega^+) \cap \overline{\mathfrak{G}}_0(\Omega^+ \cup \Gamma)$ *and* F', G' *be arbitrary* **m.v.d.f.** *from* $\mathfrak{G}_1(\Omega^+) \cap \mathfrak{G}_0(\Omega^+ \cup \Gamma)$. *Then for any* $z \in \Omega^+$:

$$\begin{aligned}
&\int\limits_{\Gamma_\zeta} F(\zeta, d\bar{z}) \underset{*}{\wedge} \overline{\boldsymbol{\sigma}}_{\zeta,z} \underset{*}{\wedge} G(\zeta, d\overline{z}) = \\
&\quad = \int\limits_{\Gamma_\zeta} F(\zeta, d\bar{z}) \underset{*}{\wedge} \overline{\boldsymbol{\sigma}}_{\zeta,z}[G](\zeta, d\overline{z}) \\
&\quad = \int\limits_{\Omega_\zeta} \overline{D}_r[F](\zeta, d\overline{z}) \underset{*}{\wedge} G(\zeta, d\bar{z}) \\
&\qquad\qquad + F(\zeta,\ d\bar{z}) \underset{\star}{\wedge} \overline{\mathcal{D}}[G](\zeta,\ d\bar{z})\, dV_\zeta, \qquad (2.8.1)
\end{aligned}$$

$$\begin{aligned}
&\int\limits_{\Gamma_\zeta} F(\zeta,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\zeta,\ z} \underset{\star}{\wedge} G(\zeta,\ d\bar{z}) = \\
&\quad = \int\limits_{\Omega^+_\zeta} \Big(\overline{\mathcal{D}}^*_r[F](\zeta,\ d\bar{z}) \underset{\star}{\wedge} G(\zeta,\ d\bar{z}) \\
&\qquad\qquad + F(\zeta,\ d\bar{z}) \underset{\star}{\wedge} \overline{\mathcal{D}}^*[G](\zeta,\ d\bar{z}) \Big)\, dV_\zeta, \qquad (2.8.2)
\end{aligned}$$

$$\begin{aligned}
&\int\limits_{\Gamma_\zeta} F'(\zeta,\ dz) \underset{\star}{\wedge} \boldsymbol{\sigma}_{\zeta,\ z} \underset{\star}{\wedge} G'(\zeta,\ dz) = \\
&\quad = \int\limits_{\Omega^+_\zeta} \Big(\mathcal{D}_r\left[F'\right](\zeta,\ dz) \underset{\star}{\wedge} G'(\zeta,\ dz) \\
&\qquad\qquad + F'(\zeta,\ dz) \underset{\star}{\wedge} \mathcal{D}\left[G'\right](\zeta,\ dz) \Big)\, dV_\zeta, \qquad (2.8.3)
\end{aligned}$$

$$\int\limits_{\Gamma_\zeta} F'(\zeta,\ dz) \underset{\star}{\wedge} \boldsymbol{\sigma}^*_{\zeta,\ z} \underset{\star}{\wedge} G'(\zeta,\ dz) =$$

$$= \int\limits_{\Omega_\zeta^+} \Big(\mathcal{D}_r^* \left[F' \right] (\zeta,\ d\mathbf{z}) \underset{\star}{\wedge} G' (\zeta,\ d\mathbf{z}) + F' (\zeta,\ d\mathbf{z}) \underset{\star}{\wedge} \mathcal{D}^* \left[G' \right] (\zeta,\ d\mathbf{z}) \Big)\, dV_\zeta. \qquad (2.8.4)$$

Proof. Apply the usual Stokes theorem and use Theorem 2.7, noting that $F (\zeta,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\zeta,\, z} \underset{\star}{\wedge} G (\zeta,\ d\bar{z})$, $F (\zeta,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\zeta,\, z} \underset{\star}{\wedge} G (\zeta,\ d\bar{z})$, $F' (\zeta,\ d\mathbf{z}) \underset{\star}{\wedge} \boldsymbol{\sigma}_{\zeta,\, z} \underset{\star}{\wedge} G' (\zeta,\ d\mathbf{z})$ and $F' (\zeta,\ d\mathbf{z}) \underset{\star}{\wedge} \boldsymbol{\sigma}^*_{\zeta,\, z} \underset{\star}{\wedge} G' (\zeta,\ d\mathbf{z})$ are well-defined on Γ. ■

2.9 The Cauchy kernel for the null-sets of the hyperholomorphic Cauchy-Riemann operators

Let us introduce the Cauchy kernels for the theory of **m.v.d.f.** from the null-sets of the corresponding operators $\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}^*$, by the formulas

$$\mathcal{K}_{\overline{\mathcal{D}}} (\zeta,\ z) := 2 \frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\zeta}_q}{|\zeta|^{2m}} \widehat{d\bar{z}^q}; & \frac{\zeta_q}{|\zeta|^{2m}} d\bar{z}^q \\ \frac{\zeta_q}{|\zeta|^{2m}} d\bar{z}^q; & \frac{\zeta_q}{|\zeta|^{2m}} \widehat{d\bar{z}^q} \end{pmatrix}, \qquad (2.9.1)$$

$$\mathcal{K}_{\overline{\mathcal{D}}^*} (\zeta,\ z) := 2 \frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\zeta_q}{|\zeta|^{2m}} d\bar{z}^q; & \frac{\bar{\zeta}_q}{|\zeta|^{2m}} \widehat{d\bar{z}^q} \\ \frac{\zeta_q}{|\zeta|^{2m}} \widehat{d\bar{z}^q}; & \frac{\zeta_q}{|\zeta|^{2m}} d\bar{z}^q \end{pmatrix}, \qquad (2.9.2)$$

and, respectively, for $\mathcal{D}$ and $\mathcal{D}^*$ by

$$\mathcal{K}_{\mathcal{D}} (\zeta,\ z) := 2 \frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\zeta_q}{|\zeta|^{2m}} \widehat{dz^q}; & \frac{\bar{\zeta}_q}{|\zeta|^{2m}} dz^q \\ \frac{\zeta_q}{|\zeta|^{2m}} dz^q; & \frac{\zeta_q}{|\zeta|^{2m}} \widehat{dz^q} \end{pmatrix}, \qquad (2.9.3)$$

$$\mathcal{K}_{\mathcal{D}^*} (\zeta,\ z) := 2 \frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\zeta}_q}{|\zeta|^{2m}} dz^q; & \frac{\zeta_q}{|\zeta|^{2m}} \widehat{dz^q} \\ \frac{\zeta_q}{|\zeta|^{2m}} \widehat{dz^q}; & \frac{\zeta_q}{|\zeta|^{2m}} dz^q \end{pmatrix}. \qquad (2.9.4)$$

$$\overline{\mathcal{D}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}$$

Note that, for any $\zeta,\ z$, $\mathcal{K}_{\overline{\mathcal{D}}}(\zeta,\ z)$ is an operator on **m.v.d.f.,** of the same type as $\overline{\boldsymbol{\sigma}}$ from Subsection 2.5 and its coefficients are of class C^∞ off the origin, which implies that for any **m.v.d.f.** F defined off the origin, $\mathcal{K}_{\overline{\mathcal{D}}}(\zeta,\ z)[F]$ is a **m.v.d.f.** and is of the same class of smoothness. It is straightforward to verify that its coefficients are harmonic functions. Note that $\overline{\mathcal{D}} \circ \mathcal{K}_{\overline{\mathcal{D}}}(\zeta,\ z)[F]$ is not identically zero, in general, off the origin, but Sections 9.6 and 9.11 explain that in a certain sense $\mathcal{K}_{\overline{\mathcal{D}}}(\zeta,\ z)$ is the fundamental solution of the Cauchy-Riemann operator $\overline{\mathcal{D}}$. The same is true for (2.9.2), (2.9.3) and (2.9.4).

Using the structure of the matrices in (2.9.1) – (2.9.4), it is not difficult to find the relations, similar to (2.3.8), between $\mathcal{K}_{\overline{\mathcal{D}}}(\zeta, z)$ and $\mathcal{K}_{\overline{\mathcal{D}}^*}(\zeta, z)$, as well as between $\mathcal{K}_{\overline{\mathcal{D}}}(\zeta, z)$ and $\mathcal{K}_{\overline{\mathcal{D}}^*}(\zeta, z)$ (see also Subsection 2.5).

2.10 Structure of the product $\mathcal{K}_{\overline{\mathcal{D}}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}$

The product mentioned in the subsection title is a very essential factor in many of the following formulas. Let us compute it. We have

$$\begin{aligned}
&\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \ = \\
&= \ 2\frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar\tau_q - \bar z_q}{|\tau - z|^{2m}} \widehat{d\bar z^q}; & \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar z^q \\ \\ \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar z^q; & \frac{\bar\tau_q - \bar z_q}{|\tau - z|^{2m}} \widehat{d\bar z^q} \end{pmatrix} \underset{\star}{\wedge} \\
&\quad \underset{\star}{\wedge} \frac{(-1)^{\frac{m(m-1)}{2}}}{(2i)^m} \sum_{j=1}^{m} (-1)^{j-1} \cdot \\
&\quad \cdot \begin{pmatrix} d\bar\tau_{[j]} \wedge d\tau \wedge d\bar z^j; & (-1)^m\, d\bar\tau \wedge d\tau_{[j]} \wedge \widehat{d\bar z^j} \\ (-1)^m\, d\bar\tau \wedge d\tau_{[j]} \wedge \widehat{d\bar z^j}; & d\bar\tau_{[j]} \wedge d\tau \wedge d\bar z^j \end{pmatrix} = \\
&= \ (-1)^{\frac{m(m-1)}{2}} 2\frac{(m-1)!}{(2\pi i)^m} \left\{ \sum_{j=1}^{m} (-1)^{j-1} \frac{\bar\tau_j - \bar z_j}{|\tau - z|^{2m}} d\bar\tau_{[j]} \wedge d\tau \wedge \widehat{d\bar z^j} \wedge \right. \\
&\qquad \left. \wedge d\bar z^j E_{2\times 2} + \right.
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^m \sum_{j=1}^{m} (-1)^{j-1} \frac{\tau_j - z_j}{|\tau - z|^{2m}} d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^j \wedge \widehat{d\bar{z}^j} E_{2\times 2} + \\
&+\sum_{q<j} \left((-1)^{j-1} \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{\tau}_{[j]} \wedge d\tau - \right. \\
&\left. - (-1)^{q-1} \frac{\tau_j - z_j}{|\tau - z|^{2m}} d\bar{\tau}_{[q]} \wedge d\tau \right) \wedge d\bar{z}^q \wedge d\bar{z}^j \breve{E}_{2\times 2} + \\
&+(-1)^m \sum_{q<j} \left((-1)^{j-1} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} d\bar{\tau} \wedge d\tau_{[j]} - \right. \\
&\left. - (-1)^{q-1} \frac{\bar{\tau}_j - \bar{z}_j}{|\tau - z|^{2m}} d\bar{\tau} \wedge d\tau_{[q]} \right) \wedge \widehat{d\bar{z}^q} \wedge \widehat{d\bar{z}^j} \breve{E}_{2\times 2} + \\
&+\sum_{q\neq j} (-1)^{j-1} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^q} \wedge d\bar{z}^j E_{2\times 2} + \\
&+(-1)^m \sum_{j\neq q} (-1)^{j-1} \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^q \wedge \\
&\quad \wedge \widehat{d\bar{z}^j} E_{2\times 2} \Big\}. \qquad (2.10.1)
\end{aligned}
$$

Introducing notations

$$
\begin{aligned}
\mathcal{U}_j(\tau,\ z) &:= \frac{(m-1)!}{(2\pi i)^m} \frac{\bar{\tau}_j - \bar{z}_j}{|\tau - z|^{2m}}, \\
\overline{\mathcal{U}}_j(\tau,\ z) &:= \frac{(m-1)!}{(2\pi i)^m} \frac{\tau_j - z_j}{|\tau - z|^{2m}},
\end{aligned}
$$

for any $j = 1,\ \ldots,\ m$, we have that

$$
\begin{aligned}
\mathcal{U}(\tau,\ z) &:= \sum_{j=1}^{m} (-1)^{j-1} \mathcal{U}_j(\tau,\ z)\, d\bar{\tau}_{[j]} \wedge d\tau = \\
&= \frac{(m-1)!}{(2\pi i)^m} \sum_{j=1}^{m} (-1)^{j-1} \frac{\bar{\tau}_j - \bar{z}_j}{|\tau - z|^{2m}} d\bar{\tau}_{[j]} \wedge d\tau, \\
\overline{\mathcal{U}}(\tau,\ z) &:= (-1)^m \sum_{j=1}^{m} (-1)^{j-1} \overline{\mathcal{U}}_j(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} = \\
&= (-1)^m \frac{(m-1)!}{(2\pi i)^m} \sum_{j=1}^{m} (-1)^{j-1} \frac{\tau_j - z_j}{|\tau - z|^{2m}} d\bar{\tau} \wedge d\tau_{[j]}.
\end{aligned}
$$

$$\overline{\mathcal{D}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}$$

The first formula gives the well-known Bochner-Martinelli kernel for holomorphic functions, which is why we will call the second kernel, $\overline{\mathcal{U}}(\tau,\ z)$, the Bochner-Martinelli kernel for antiholomorphic functions. Note that $\overline{\mathcal{U}}$ is not only a notation but also the complex conjugate to $\mathcal{U}$. Extending the idea we introduce

$$\begin{aligned}
\mathcal{U}_{\mathbf{j}}(\tau,\ z) \ := \ & \frac{(m-1)!}{(2\pi i)^m}\left(\sum_{\substack{p=1\\ p\neq j_1,\ldots,j_k}}^{m}(-1)^{p-1}\frac{\bar{\tau}_p-\bar{z}_p}{|\tau-z|^{2m}}d\bar{\tau}_{[p]}\wedge d\tau+\right.\\
& \left.+(-1)^m\sum_{p=j_1,\ldots,j_k}(-1)^{p-1}\frac{\tau_p-z_p}{|\tau-z|^{2m}}d\bar{\tau}\wedge d\tau_{[p]}\right)\\
= \ & \sum_{\substack{p=1\\ p\neq j_1,\ldots,j_k}}^{m}(-1)^{p-1}\mathcal{U}_p(\tau,z)\,d\bar{\tau}_{[p]}\wedge d\tau+\\
& +(-1)^m\sum_{p=j_1,\ldots,j_k}(-1)^{p-1}\overline{\mathcal{U}}_p(\tau,z)\,d\bar{\tau}\wedge d\tau_{[p]}
\end{aligned} \tag{2.10.2}$$

for any strictly increasing $|\mathbf{j}|$-tuple $\mathbf{j}$ in $\{1,\ \ldots,\ m\}$, including $\mathbf{j}=\emptyset$. We will see later that $\mathcal{U}_{\mathbf{j}}(\tau,\ z)$ plays the same role for functions antiholomorphic in z_{j_1}, …, z_{j_k} and holomorphic in the rest variables that $\mathcal{U}(\tau,\ z)$ plays for holomorphic functions. Under these notations, we have

$$\begin{aligned}
&\mathcal{K}_{\overline{\mathcal{D}}}(\tau-z,\ z)\underset{\star}{\wedge}\overline{\boldsymbol{\sigma}}_{\tau,\ z}=\\
&= \ (-1)^{\frac{m(m-1)}{2}}2\left\{\sum_{j=1}^{m}(-1)^{j-1}\mathcal{U}_j(\tau,\ z)\,d\bar{\tau}_{[j]}\wedge d\tau\wedge\widehat{d\bar{z}^j}\wedge\right.\\
&\quad \wedge d\bar{z}^j E_{2\times 2}+\\
&\quad +(-1)^m\sum_{j=1}^{m}(-1)^{j-1}\overline{\mathcal{U}}_j(\tau,\ z)\,d\bar{\tau}\wedge d\tau_{[j]}\wedge d\bar{z}^j\wedge\widehat{d\bar{z}^j}E_{2\times 2}+\\
&\quad \left.+\sum_{q<j}\Big((-1)^{j-1}\overline{\mathcal{U}}_q(\tau,\ z)\,d\bar{\tau}_{[j]}\wedge d\tau-\right.
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{q-1} \overline{\mathcal{U}}_j\,(\tau,\ z)\, d\bar{\tau}_{[q]} \wedge d\tau \Big) \wedge d\bar{z}^q \wedge d\bar{z}^j \breve{E}_{2\times 2} + \\
&+ (-1)^m \sum_{q<j} \Big((-1)^{j-1} \mathcal{U}_q\,(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} - \\
&- (-1)^{q-1} \mathcal{U}_j\,(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[q]} \Big) \wedge \widehat{d\bar{z}^q} \wedge \widehat{d\bar{z}^j} \breve{E}_{2\times 2} + \\
&+ \sum_{q\neq j} (-1)^{j-1} \mathcal{U}_q\,(\tau,\ z)\, d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^q} \wedge d\bar{z}^j E_{2\times 2} + \\
&+ (-1)^m \sum_{j\neq q} (-1)^{j-1} \overline{\mathcal{U}}_q\,(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^q \wedge \widehat{d\bar{z}^j} E_{2\times 2} \Bigg\}.
\end{aligned}
$$

This expression, $\mathcal{K}_{\overline{\mathcal{D}}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}$, defines an operator acting on $\overline{\mathfrak{G}}_0(\Omega \setminus\{z\})$. In particular on the set $\overline{\mathfrak{G}}_0^0\,(\Omega \setminus \{z\})$ of matrices whose entries are scalar-valued functions, it takes the form

$$
\begin{aligned}
&\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} = \\
&= \ (-1)^{\frac{m(m-1)}{2}} 2 \Big\{ \mathcal{U}(\zeta, z) E_{2\times 2} + \sum_{q<j} ((-1)^{j-1} \overline{\mathcal{U}}_q(\tau, z) d\bar{\tau}_{[j]} \wedge d\tau - \\
&\quad -(-1)^{q-1} \overline{\mathcal{U}}_j \tau, z) d\bar{\tau}_{[q]} \wedge d\tau) \wedge d\bar{z}^q \wedge d\bar{z}^j \breve{E}_{2\times 2} \Big\}.
\end{aligned}
$$

Let Γ be a real $(2m-1)$-surface of class C^1. Taking into account the contents of Subsection 2.5 we obtain, for any $F \in \overline{\mathfrak{G}}_0\,(\Omega \setminus \{z\}) \cap \overline{\mathfrak{G}}_0\,(\Omega \cup \Gamma)$,

$$
\begin{aligned}
&\Big\{ \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} F\,(\tau,\ d\bar{z}) \Big\} |\Gamma = \\
&= \ (2i)^m \Bigg\{ \sum_{j=1}^m \mathcal{U}_j\,(\tau,\ z)\, n_{j;\,\tau} \widehat{d\bar{z}^j} \wedge d\bar{z}^j \wedge F\,(\tau,\ d\bar{z}) + \\
&\quad + \sum_{j=1}^m \overline{\mathcal{U}}_j\,(\tau,\ z)\, \bar{n}_{j;\,\tau} d\bar{z}^j \wedge \widehat{d\bar{z}^j} \wedge F\,(\tau,\ d\bar{z}) +
\end{aligned}
$$

$$+\sum_{q<j}\left(\overline{\mathcal{U}}_q\left(\tau,\ z\right)n_{j;\,\tau}-\overline{\mathcal{U}}_j\left(\tau,\ z\right)n_{q;\,\tau}\right)\wedge$$
$$\wedge d\bar{z}^q\wedge d\bar{z}^j\breve{E}_{2\times 2}\underset{\star}{\wedge}F\left(\tau,\ d\bar{z}\right)+$$
$$+\sum_{q<j}\left(\mathcal{U}_q\left(\tau,\ z\right)\bar{n}_{j;\,\tau}-\mathcal{U}_j\left(\tau,\ z\right)\bar{n}_{q,\,\tau}\right)\wedge$$
$$\wedge\widehat{d\bar{z}^q}\wedge\widehat{d\bar{z}^j}\breve{E}_{2\times 2}\underset{\star}{\wedge}F\left(\tau,\ d\bar{z}\right)+$$
$$+\sum_{q\neq j}\mathcal{U}_q\left(\tau,\ z\right)n_{j,\,\tau}\widehat{d\bar{z}^q}\wedge d\bar{z}^j\wedge F\left(\tau,\ d\bar{z}\right)+$$
$$\left.+\sum_{j\neq q}\overline{\mathcal{U}}_q\left(\tau,\ z\right)\bar{n}_{j;\,\tau}d\bar{z}^q\wedge\widehat{d\bar{z}^j}\wedge F\left(\tau,\ d\bar{z}\right)\right\}dS_\tau;$$

and for $F\in\overline{\mathfrak{G}}_0^0\left(\Omega\setminus\{z\}\right)\cap\overline{\mathfrak{G}}_0^0\left(\Omega\cup\Gamma\right)$,

$$\left\{\mathcal{K}_{\overline{\mathcal{D}}}\left(\tau-z,\ z\right)\underset{\star}{\wedge}\overline{\boldsymbol{\sigma}}_{\tau,\,z}\underset{\star}{\wedge}F\left(\tau,\ d\bar{z}\right)\right\}|\Gamma=$$
$$=\ (2i)^m\cdot\begin{pmatrix}\phi_{11}&\phi_{12}\\ \phi_{21}&\phi_{22}\end{pmatrix}\underset{\star}{\wedge}F\left(\tau,\ d\bar{z}\right)dS_\tau$$

where

$$\phi_{11}=\sum_{j=1}^m\mathcal{U}_j\left(\tau,\ z\right)n_{j;\,\tau};$$
$$\phi_{12}=\sum_{q<j}\overline{\mathcal{U}}_q\left(\tau,\ z\right)n_{j;\,\tau}d\bar{z}^q\wedge d\bar{z}^j-\sum_{q<j}\overline{\mathcal{U}}_j\left(\tau,\ z\right)n_{q;\tau}d\bar{z}^q\wedge d\bar{z}^j;$$
$$\phi_{21}=\sum_{q<j}\overline{\mathcal{U}}_q\left(\tau,\ z\right)n_{j;\,\tau}d\bar{z}^q\wedge d\bar{z}^j-\sum_{q<j}\overline{\mathcal{U}}_j\left(\tau,\ z\right)n_{q;\,\tau}d\bar{z}^q\wedge d\bar{z}^j;$$
$$\phi_{22}=\sum_{j=1}^m\mathcal{U}_j\left(\tau,\ z\right)n_{j;\,\tau}.$$

2.11 Borel-Pompeiu (or Cauchy-Green) formula for smooth differential matrix-forms

Theorem *Let Ω^+ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $F\in\overline{\mathfrak{G}}_1\left(\Omega^+\right)\cap\overline{\mathfrak{G}}\left(\Omega^+\cup\Gamma\right)$ and*

$G \in \mathfrak{G}_1(\Omega^+) \cap \mathfrak{G}(\Omega^+ \cup \Gamma)$. *Then the following equalities hold in* Ω^+*:*

$$\begin{aligned} 2F(z) &= \int_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) - \\ &\quad - \int_{\Omega^+} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}[F](\tau,\ d\bar{z})\, dV_\tau, \end{aligned} \tag{2.11.1}$$

$$\begin{aligned} 2G(z) &= \int_{\Gamma} \mathcal{K}_{\mathcal{D}}(\tau - z,\ z) \underset{\star}{\wedge} \boldsymbol{\sigma}_{\tau,\, z} \underset{\star}{\wedge} G(\tau,\ dz) - \\ &\quad - \int_{\Omega^+} \mathcal{K}_{\mathcal{D}}(\tau - z,\ z) \underset{\star}{\wedge} \mathcal{D}[G](\tau,\ dz)\, dV_\tau. \end{aligned} \tag{2.11.2}$$

Proof. The proof will be given for the first case only. Take $z \in \Omega^+$ fixed and choose $\epsilon > 0$ such that $\overline{\mathbb{B}}(z;\ \epsilon) \subset \Omega^+$.

By Stokes formula we have

$$\begin{aligned} &\int_{Fr\left(\Omega^+ \setminus \overline{\mathbb{B}}(z;\ \epsilon)\right)} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) = \\ &= \int_{\Omega^+ \setminus \overline{\mathbb{B}}(z;\ \epsilon)} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}[F](\tau,\ d\bar{z})\, dV_\tau. \end{aligned} \tag{2.11.3}$$

As $\overline{\mathcal{D}}[F]$ is continuous in Ω^+ and Ω^+ is a bounded set,

$$\mathcal{K}_{\overline{\mathcal{D}}}(\tau\ -\ z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}[F](\tau,\ d\bar{z})\, dV_\tau$$

is Lebesgue absolutely integrable on Ω^+. Consequently, by taking the limit for $\epsilon \to 0^+$ we get in the right-hand side of (2.11.3):

$$\int_{\Omega^+} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}[F](\tau,\ d\bar{z})\, dV_\tau.$$

As to the left-hand side of (2.11.3), it can be put into the form

$$\int\limits_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) -$$

$$- \int\limits_{\mathbb{S}(z;\ \epsilon)} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}).$$

We have

$$\int\limits_{\mathbb{S}(z;\ \epsilon)} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) =$$

$$= \frac{(m-1)!}{\pi^m} \int\limits_{\mathbb{S}(z;\ \epsilon)} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q}; & \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q \\ \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q; & \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q} \end{pmatrix} \underset{\star}{\wedge}$$

$$\underset{\star}{\wedge} \sum_{j=1}^{m} \begin{pmatrix} n_j d\bar{z}^j; & \bar{n}_j \widehat{d\bar{z}^j} \\ \bar{n}_j \widehat{d\bar{z}^j}; & n_j d\bar{z}^j \end{pmatrix} \underset{\star}{\wedge} F(\tau,\ d\bar{z})\, dS_\tau.$$

Since for the sphere, $n_j = \frac{1}{\epsilon}(\tau_j - z_j)$ and $\bar{n}_j = \frac{1}{\epsilon}(\bar{\tau}_j - \bar{z}_j)$, we obtain that

$$\int\limits_{\mathbb{S}(z;\ \epsilon)} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) =$$

$$= \frac{(m-1)!}{\pi^m} \int\limits_{\mathbb{S}(z;\epsilon)} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon^{2m}} \widehat{d\bar{z}^q} & \frac{\tau_q - z_q}{\epsilon^{2m}} d\bar{z}^q \\ \frac{\tau_q - z_q}{\epsilon^{2m}} d\bar{z}^q & \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon^{2m}} \widehat{d\bar{z}^q} \end{pmatrix} \underset{\star}{\wedge}$$

$$\underset{\star}{\wedge} \sum_{j=1}^{m} \begin{pmatrix} \frac{\tau_j - z_j}{\epsilon} d\bar{z}^j & \frac{\bar{\tau}_j - \bar{z}_j}{\epsilon} \widehat{d\bar{z}^j} \\ \frac{\bar{\tau}_j - \bar{z}_j}{\epsilon} \widehat{d\bar{z}^j} & \frac{\tau_j - z_j}{\epsilon} d\bar{z}^j \end{pmatrix} \underset{\star}{\wedge} F(\tau,\ d\bar{z})\, dS_\tau =$$

$$= \frac{(m-1)!}{\pi^m} \int\limits_{\mathbb{S}(z;\ \epsilon)} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon^{2m}} \widehat{d\bar{z}^q} & \frac{\tau_q - z_q}{\epsilon^{2m}} d\bar{z}^q \\ \frac{\tau_q - z_q}{\epsilon^{2m}} d\bar{z}^q & \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon^{2m}} \widehat{d\bar{z}^q} \end{pmatrix} \underset{\star}{\wedge}$$

$$\underset{\star}{\wedge}\begin{pmatrix} \frac{\tau_q - z_q}{\epsilon} d\bar{z}^q & \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon}\widehat{d\bar{z}^q} \\ \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon}\widehat{d\bar{z}^q} & \frac{\tau_q - z_q}{\epsilon} d\bar{z}^q \end{pmatrix} \underset{\star}{\wedge} F\left(\tau,\ d\bar{z}\right) dS_\tau +$$

$$+\frac{(m-1)!}{\pi^m} \int\limits_{\mathbb{S}(z;\,\epsilon)} \sum_{q \neq j} \begin{pmatrix} \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon^{2m}}\widehat{d\bar{z}^q} & \frac{\tau_q - z_q}{\epsilon^{2m}} d\bar{z}^q \\ \frac{\tau_q - z_q}{\epsilon^{2m}} d\bar{z}^q & \frac{\bar{\tau}_q - \bar{z}_q}{\epsilon^{2m}}\widehat{d\bar{z}^q} \end{pmatrix} \underset{\star}{\wedge}$$

$$\underset{\star}{\wedge}\begin{pmatrix} \frac{\tau_j - z_j}{\epsilon} d\bar{z}^j & \frac{\bar{\tau}_j - \bar{z}_j}{\epsilon}\widehat{d\bar{z}^j} \\ \frac{\bar{\tau}_j - \bar{z}_j}{\epsilon}\widehat{d\bar{z}^j} & \frac{\tau_j - z_j}{\epsilon} d\bar{z}^j \end{pmatrix} \underset{\star}{\wedge} F\left(\tau,\ d\bar{z}\right) dS_\tau.$$

For the first integral we have

$$\frac{(m-1)!}{\pi^m}\frac{1}{\epsilon^{2m+1}} \int\limits_{\mathbb{S}(z;\,\epsilon)} \sum_{q=1}^{m} \left(\left|\bar{\tau}_q - \bar{z}_q\right|^2 \widehat{d\bar{z}^q} \wedge d\bar{z}^q + \right.$$

$$\left. + \left|\tau_q - z_q\right|^2 d\bar{z}^q \wedge \widehat{d\bar{z}^q} \right) \cdot E_{2\times 2} \underset{\star}{\wedge} F\left(\tau,\ d\bar{z}\right) dS_\tau \quad =$$

$$= \quad \frac{(m-1)!}{\pi^m}\frac{1}{\epsilon^{2m-1}} \int\limits_{\mathbb{S}(z;\,\epsilon)} F\left(\tau,\ d\bar{z}\right) dS_\tau =$$

$$= \quad \frac{(m-1)!}{\pi^m}\frac{1}{\epsilon^{2m-2}} \int\limits_{\mathbb{S}(z;\,\epsilon)} \frac{F\left(\tau,\ d\bar{z}\right) - F\left(z\right)}{\epsilon} dS_\tau +$$

$$+\frac{(m-1)!}{\pi^m}\frac{1}{\epsilon^{2m-1}} \int\limits_{\mathbb{S}(z;\,\epsilon)} F\left(z\right) dS_\tau =$$

$$= \quad \frac{(m-1)!}{\pi^m}\frac{1}{\epsilon^{2m-2}} \int\limits_{\mathbb{S}(z;\,\epsilon)} \frac{F\left(\tau,\ d\bar{z}\right) - F\left(z\right)}{\epsilon} dS_\tau + 2F\left(z\right),$$

where

$$\frac{1}{\epsilon^{2m-2}} \left| \int\limits_{\mathbb{S}(z;\,\epsilon)} \frac{F\left(\tau,\ d\bar{z}\right) - F\left(z\right)}{\epsilon} dS_\tau \right| \leq \frac{1}{\epsilon^{2m-2}} \mathrm{const} \int\limits_{\mathbb{S}(z;\,\epsilon)} dS_\tau \leq$$

$$\leq \text{const} \cdot \epsilon.$$

Hence the first integral tends to $2F(z)$ when ϵ tends to zero. Now, using the same idea for the second integral, we see that it tends to zero iff the following integral tends to zero when ϵ tends to zero:

$$\int\limits_{\mathbb{S}(z;\,\epsilon)} \sum_{q\neq j} \begin{pmatrix} \frac{\bar\tau_q-\bar z_q}{\epsilon^{2m}}\widehat{d\bar z^q} & \frac{\tau_q-z_q}{\epsilon^{2m}}d\bar z^q \\ \frac{\tau_q-z_q}{\epsilon^{2m}}d\bar z^q & \frac{\bar\tau_q-\bar z_q}{\epsilon^{2m}}\widehat{d\bar z^q} \end{pmatrix} \underset{\star}{\wedge} \begin{pmatrix} \frac{\tau_j-z_j}{\epsilon}d\bar z^j & \frac{\bar\tau_j-\bar z_j}{\epsilon}\widehat{d\bar z^j} \\ \frac{\bar\tau_j-\bar z_j}{\epsilon}\widehat{d\bar z^j} & \frac{\tau_j-z_j}{\epsilon}d\bar z^j \end{pmatrix} \underset{\star}{\wedge}$$

$$\underset{\star}{\wedge} E_{2\times 2}\, dS_\tau.$$

To prove the last identity it is sufficient to prove that, for any $j \neq q$,

$$\int\limits_{\mathbb{S}(z;\,\epsilon)} (\bar\tau_q - \bar z_q)(\bar\tau_j - \bar z_j)\, dS_\tau = 0, \tag{2.11.4}$$

$$\int\limits_{\mathbb{S}(z;\,\epsilon)} (\bar\tau_q - \bar z_q)(\tau_j - z_j)\, dS_\tau = 0. \tag{2.11.5}$$

Let us make a change of variables:

$$\begin{aligned}
x_1 = Re\,(z_1) &= \epsilon \cos\theta_{2m-1} \cdots \cos\theta_2 \cos\theta_1, \\
y_1 = Im\,(z_1) &= \epsilon \cos\theta_{2m-1} \cdots \cos\theta_2 \sin\theta_1, \\
x_2 = Re\,(z_2) &= \epsilon \cos\theta_{2m-1} \cdots \cos\theta_3 \sin\theta_2, \\
y_2 = Im\,(z_2) &= \epsilon \cos\theta_{2m-1} \cdots \cos\theta_4 \sin\theta_3, \\
\cdots\cdots\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\
x_m = Re\,(z_m) &= \epsilon \cos\theta_{2m-1} \sin\theta_{2m-2}, \\
y_m = Im\,(z_m) &= \epsilon \sin\theta_{2m-1},
\end{aligned}$$

where $0 \leq \theta_1 \leq 2\pi$, $-\frac{\pi}{2} \leq \theta_i \leq \frac{\pi}{2}$, $i = 2,\ 3,\ \ldots,\ 2m-1$. To obtain (2.11.4)–(2.11.5) it is sufficient to prove that the following usual integrals are equal to zero: for any $2 < j \leq q$,

$$\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ldots \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta_{2m-1} \ldots \cos^2\theta_q \sin\theta_{q-1} \cos\theta_{q-1} \ldots$$

$$\ldots \cos\theta_j \sin\theta_{j-1} d\theta \;=\; 0,$$

for any $2 < q$,

$$\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dots \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{0}^{2\pi} \cos^2\theta_{2m-1} \dots \cos^2\theta_q \sin\theta_{q-1} \cos\theta_{q-1} \dots \\ \dots \cos\theta_2 \sin\theta_1 d\theta = 0,$$

and for any $1 < q$,

$$\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dots \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{0}^{2\pi} \cos^2\theta_{2m-1} \dots \cos^2\theta_q \sin\theta_{q-1} \cos\theta_{q-1} \dots \\ \dots \cos\theta_2 \cos\theta_1 d\theta = 0,$$

which is trivially true.

Hence, finally,

$$\lim_{\epsilon \to 0^+} \int\limits_{\mathbb{S}(z;\, \epsilon)} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) = 2F(z). \blacksquare$$

2.11.1 Structure of the Borel-Pompeiu formula

First consider the product $\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}[F](\tau,\ d\bar{z})$. We have

$$\begin{aligned} &\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}[F](\tau,\ d\bar{z}) = \\ &= 2\frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q}; & \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q \\ \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q; & \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q} \end{pmatrix} \underset{\star}{\wedge} \Lambda(\tau,\ d\bar{z}) = \\ &=: R = \left(R^{ij}\right), \end{aligned}$$

where

$$\Lambda(\tau,\ d\bar{z}) = (\lambda_{pq})_{p,q=1}^{2},$$

$$\lambda_{11} = \bar{\partial}\left[F^{11}\right](\tau,\ d\bar{z}) + \bar{\partial}^*\left[F^{21}\right](\tau,\ d\bar{z});$$

$$\lambda_{12} = \bar{\partial}\left[F^{12}\right](\tau,\ d\bar{z}) + \bar{\partial}^*\left[F^{22}\right](\tau,\ d\bar{z});$$

$$\lambda_{21} = \bar{\partial}^*\left[F^{11}\right](\tau,\ d\bar{z}) + \bar{\partial}\left[F^{21}\right](\tau,\ d\bar{z});$$

$$\lambda_{22} \quad = \quad \bar{\partial}^* \left[F^{12}\right](\tau,\ d\bar{z}) + \bar{\partial}\left[F^{22}\right](\tau,\ d\bar{z})\,;$$

and

$$\begin{aligned}
R_{11} \quad = \quad & 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}}\widehat{d\bar{z}^q} \wedge \left(\bar{\partial}\left[F^{11}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}^*\left[F_{21}\right](\tau,\ d\bar{z})\right) + \\
& + 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q - z_q}{|\tau - z|^{2m}}d\bar{z}^q \wedge \left(\bar{\partial}^*\left[F^{11}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}\left[F^{21}\right](\tau,\ d\bar{z})\right),
\end{aligned}$$

$$\begin{aligned}
R_{21} \quad = \quad & 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q - z_q}{|\tau - z|^{2m}}d\bar{z}^q \wedge \left(\bar{\partial}\left[F^{11}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}^*\left[F^{21}\right](\tau,\ d\bar{z})\right) + \\
& + 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}}\widehat{d\bar{z}^q} \wedge \left(\bar{\partial}^*\left[F^{11}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}\left[F^{21}\right](\tau,\ d\bar{z})\right),
\end{aligned}$$

$$\begin{aligned}
R_{12} \quad = \quad & 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}}\widehat{d\bar{z}^q} \wedge \left(\bar{\partial}\left[F^{12}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}^*\left[F_{22}\right](\tau,\ d\bar{z})\right) + \\
& + 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q - z_q}{|\tau - z|^{2m}}d\bar{z}^q \wedge \left(\bar{\partial}^*\left[F^{12}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}\left[F^{22}\right](\tau,\ d\bar{z})\right),
\end{aligned}$$

$$\begin{aligned}
R_{22} \quad = \quad & 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q - z_q}{|\tau - z|^{2m}}d\bar{z}^q \wedge \left(\bar{\partial}\left[F^{12}\right](\tau,\ d\bar{z}) + \right. \\
& \left. + \bar{\partial}^*\left[F^{22}\right](\tau,\ d\bar{z})\right) + \\
& + 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}}\widehat{d\bar{z}^q} \wedge \left(\bar{\partial}^*\left[F^{12}\right](\tau,\ d\bar{z}) + \right.
\end{aligned}$$

$$+\bar{\partial}\left[F^{22}\right](\tau,\ d\bar{z})\Big).$$

Let us now substitute all this into the formula (2.11.1). We have

$$\begin{aligned}
2F^{\alpha\beta}(z) = \\
= \ & (-1)^{\frac{m(m-1)}{2}} 2 \cdot \\
& \cdot \int_{\Gamma} \Bigg\{ \sum_{j=1}^{m} (-1)^{j-1} \mathcal{U}_j(\tau,\ z)\, d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^j} \wedge d\bar{z}^j \wedge \\
& \qquad \wedge F^{\alpha\beta}(\tau,\ d\bar{z}) + \\
& + (-1)^m \sum_{j=1}^{m} (-1)^{j-1} \overline{\mathcal{U}}_j(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^j \wedge \widehat{d\bar{z}^j} \wedge \\
& \qquad \wedge F^{\alpha\beta}(\tau, d\bar{z}) + \\
& + \sum_{q \neq j} (-1)^{j-1} \mathcal{U}_q(\tau,\ z)\, d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^q} \wedge d\bar{z}^j \wedge F^{\alpha\beta}(\tau,\ d\bar{z}) + \\
& + (-1)^m \sum_{q \neq j} (-1)^{j-1} \overline{\mathcal{U}}_q(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^q \wedge \widehat{d\bar{z}^j} \wedge \\
& \qquad \wedge F^{\alpha\beta}(\tau, d\bar{z}) \Bigg\} - \\
& - 2(2i)^m \int_{\Omega^+} \Bigg\{ \sum_{j=1}^{m} \mathcal{U}_j(\tau,\ z)\, \widehat{d\bar{z}^j} \wedge d\bar{z}^j \wedge \frac{\partial F^{\alpha\beta}}{\partial \bar{z}_j}(\tau,\ d\bar{z}) + \\
& + \sum_{j=1}^{m} \overline{\mathcal{U}}_j(\tau,\ z)\, d\bar{z}^j \wedge \widehat{d\bar{z}^j} \wedge \frac{\partial F^{\alpha\beta}}{\partial z_j}(\tau,\ d\bar{z}) + \\
& + \sum_{q \neq j} \mathcal{U}_q(\tau,\ z)\, \widehat{d\bar{z}^q} \wedge d\bar{z}^j \wedge \frac{\partial F^{\alpha\beta}}{\partial \bar{z}_j}(\tau,\ d\bar{z}) + \\
& + \sum_{q \neq j} \overline{\mathcal{U}}_q(\tau,\ z)\, d\bar{z}^q \wedge \widehat{d\bar{z}^j} \wedge \frac{\partial F^{\alpha\beta}}{\partial z_j}(\tau,\ d\bar{z}) \Bigg\} dV_\tau, \qquad (2.11.6)
\end{aligned}$$

$$0 \ = \ (-1)^{\frac{m(m-1)}{2}} 2 \int_{\Gamma} \Bigg\{ \sum_{q<j} \Big((-1)^{j-1} \overline{\mathcal{U}}_q(\tau, z)\, d\bar{\tau}_{[j]} \wedge d\tau \ -$$

$$
\begin{aligned}
&- (-1)^{q-1}\,\overline{\mathcal{U}}_j\,(\tau,\ z)\,d\bar{\tau}_{[q]}\wedge d\tau\Big)\wedge d\bar{z}^q\wedge d\bar{z}^j\wedge F^{\alpha\beta}\,(\tau,\ d\bar{z})+\\
&+(-1)^m\sum_{q<j}\Big((-1)^{j-1}\mathcal{U}_q\,(\tau,\ z)\,d\bar{\tau}\wedge d\tau_{[j]}\ -\\
&-(-1)^{q-1}\,\mathcal{U}_j\,(\tau,\ z)\,d\bar{\tau}\wedge d\tau_{[q]}\Big)\wedge\widehat{d\bar{z}^q}\wedge\widehat{d\bar{z}^j}\wedge F^{\alpha\beta}\,(\tau,\ d\bar{z})\Big\}-\\
&-2\,(2i)^m\int\limits_{\Omega^+}\left\{\sum_{q\neq j}\overline{\mathcal{U}}_q\,(\tau,\ z)\,d\bar{z}^q\wedge d\bar{z}^j\wedge\frac{\partial F^{\alpha\beta}}{\partial\bar{z}_j}\,(\tau,\ d\bar{z})+\right.\\
&\left.+\sum_{q\neq j}\mathcal{U}_q\,(\tau,\ z)\,\widehat{d\bar{z}^q}\wedge\widehat{d\bar{z}^j}\wedge\frac{\partial F^{\alpha\beta}}{\partial z_j}\,(\tau,\ d\bar{z})\right\}dV_\tau.
\end{aligned}
\tag{2.11.7}
$$

Thus we arrived at an integral representation of a smooth differential form expressed in terms of the Bochner-Martinelli-like kernels, together with a certain identity. Let us analyze them more rigorously, starting naturally with the case of one variable.

2.11.2 The case $m=1$

Let $F=\begin{pmatrix}\varphi^{11}&\varphi^{12}\\ \varphi^{21}&\varphi^{22}\end{pmatrix}\in\overline{\mathfrak{G}}_1^0$. Then

$$
\begin{aligned}
2F\,(z) &= \frac{1}{\pi i}\int\limits_{\Gamma}\begin{pmatrix}\frac{\varphi^{11}d\tau}{\tau-z}&\frac{\varphi^{12}d\tau}{\tau-z}\\ \frac{\varphi^{21}d\tau}{\tau-z}&\frac{\varphi^{22}d\tau}{\tau-z}\end{pmatrix}-\\
&\quad-\frac{2}{\pi}\int\limits_{\Omega^+}\begin{pmatrix}\frac{1}{\tau-z}\frac{\partial\varphi^{11}}{\partial\bar{z}}&\frac{1}{\tau-z}\frac{\partial\varphi^{12}}{\partial\bar{z}}\\ \frac{1}{\tau-z}\frac{\partial\varphi^{21}}{\partial\bar{z}}&\frac{1}{\tau-z}\frac{\partial\varphi^{22}}{\partial\bar{z}}\end{pmatrix}dV_\tau=\\
&= \frac{1}{\pi i}\int\limits_{\Gamma}\frac{F\,(\tau)\,d\tau}{\tau-z}-\frac{2}{\pi}\int\limits_{\Omega^+}\frac{1}{\tau-z}\frac{\partial F}{\partial\bar{z}}\,(\tau)\,dV_\tau.
\end{aligned}
\tag{2.11.8}
$$

As a matter of fact, the above equality with the matrix F dissolves into a system of four independent equalities of the same form

(2.11.8), each one with its own function $\varphi^{\alpha\beta}$. That is, (2.11.8) is equivalent to the four Borel-Pompeiu formulas from the usual one-dimensional complex analysis, with the holomorphic Cauchy kernel.

Now for $F = \begin{pmatrix} \psi_1^{11} d\bar{z} & \psi_1^{12} d\bar{z} \\ \psi_1^{21} d\bar{z} & \psi_1^{22} d\bar{z} \end{pmatrix} \in \overline{\mathfrak{G}}_1^1$, we have

$$\begin{aligned} 2F(z) &= -\frac{1}{\pi i} \int\limits_{\Gamma} \begin{pmatrix} \frac{\psi_1^{11} d\bar{\tau}}{\bar{\tau}-\bar{z}} d\bar{z} & \frac{\psi_1^{12} d\bar{\tau}}{\bar{\tau}-\bar{z}} d\bar{z} \\ \frac{\psi_1^{21} d\bar{\tau}}{\bar{\tau}-\bar{z}} d\bar{z} & \frac{\psi_1^{22} d\bar{\tau}}{\bar{\tau}-\bar{z}} d\bar{z} \end{pmatrix} - \\ &\quad -\frac{2}{\pi} \int\limits_{\Omega^+} \begin{pmatrix} \frac{1}{\bar{\tau}-\bar{z}} \frac{\partial \psi_1^{11}}{\partial z} d\bar{z} & \frac{1}{\bar{\tau}-\bar{z}} \frac{\partial \psi_1^{12}}{\partial z} d\bar{z} \\ \frac{1}{\bar{\tau}-\bar{z}} \frac{\partial \psi_1^{21}}{\partial z} d\bar{z} & \frac{1}{\bar{\tau}-\bar{z}} \frac{\partial \psi_1^{22}}{\partial z} d\bar{z} \end{pmatrix} dV_\tau = \\ &= -\frac{1}{\pi i} \int\limits_{\Gamma} \frac{F(\tau,\ d\bar{z})\, d\bar{\tau}}{\bar{\tau}-\bar{z}} - \frac{2}{\pi} \int\limits_{\Omega^+} \frac{1}{\bar{\tau}-\bar{z}} \frac{\partial F}{\partial z}(\tau,\ d\bar{z})\, dV_\tau. \end{aligned}$$

This means that we arrived at the Borel-Pompeiu formula in one variable but with the antiholomorphic Cauchy kernel. Notice also that the identity (2.11.7) does not give any information.

2.11.3 The case $m = 2$

We consider here, again, different types of differential forms. First of all, for functions, that is, on $\overline{\mathfrak{G}}_1^0$, we have

$$\begin{aligned} 2F(z) &= -2 \int\limits_{\Gamma} \mathcal{U}(\tau,\ z)\, F(\tau) - \\ &\quad -2 \frac{1}{\pi^2} \int\limits_{\Omega^+} \sum_{j=1}^{m} \frac{\bar{\tau}_j - \bar{z}_j}{|\tau - z|^4} \frac{\partial F}{\partial \bar{z}_j}(\tau)\, dV_\tau, \\ 0 &= -2 \frac{1}{(2\pi i)^2} \int\limits_{\Gamma} \left(-\frac{\tau_1 - z_1}{|\tau - z|^4} d\bar{\tau}^1 \wedge d\tau - \right. \\ &\quad \left. -\frac{\tau_2 - z_2}{|\tau - z|^4} d\bar{\tau}^2 \wedge d\tau \right) F(\tau) - \end{aligned}$$

$$- 2\frac{1}{\pi^2}\int\limits_{\Omega^+}\left(\frac{\tau_1 - z_1}{|\tau - z|^4}\frac{\partial F}{\partial \bar{z}_2}(\tau) - \frac{\tau_2 - z_2}{|\tau - z|^4}\frac{\partial F}{\partial \bar{z}_1}(\tau)\right) dV_\tau,$$

where $F = \begin{pmatrix} \varphi^{11} & \varphi^{12} \\ \varphi^{21} & \varphi^{22} \end{pmatrix}$.

The first equality is nothing more than the Borel-Pompeiu formula for the Bochner-Martinelli kernel and for $m = 2$, see [AiYu], [HL1], [Ky] and many others. Note that in both formulas, the volume integrals disappear for holomorphic $\varphi^{\alpha\beta}$.

For $F = \begin{pmatrix} \psi_2^{11} d\bar{z}^2 & \psi_2^{12} d\bar{z}^2 \\ \\ \psi_2^{21} d\bar{z}^2 & \psi_2^{22} d\bar{z}^2 \end{pmatrix}$ of class C^1, we have:

$$\begin{aligned}
2F(z) &= -2\int\limits_{\Gamma}\mathcal{U}_{(2)}(\tau,\ z)\,F(\tau,\ d\bar{z}) - \\
&\quad - 2\frac{1}{\pi^2}\int\limits_{\Omega^+}\left(\frac{\bar{\tau}_1 - \bar{z}_1}{|\tau - z|^4}\frac{\partial F}{\partial \bar{z}_1}(\tau,\ d\bar{z}) + \right. \\
&\quad \left. + \frac{\tau_2 - z_2}{|\tau - z|^4}\frac{\partial F}{\partial z_2}(\tau,\ d\bar{z})\right) dV_\tau, \\
0 &= 2\frac{1}{(2\pi i)^2}\int\limits_{\Gamma}\left(\frac{\tau_1 - z_1}{|\tau - z|^4} d\bar{\tau}\wedge d\tau^1 + \right. \\
&\quad \left. + \frac{\bar{\tau}_2 - \bar{z}_2}{|\tau - z|^4} d\bar{\tau}^2\wedge d\tau\right) F(\tau,\ d\bar{z}) - \\
&\quad - 2\frac{1}{\pi^2}\int\limits_{\Omega^+}\left(\frac{\tau_1 - z_1}{|\tau - z|^4}\frac{\partial F}{\partial z_2}(\tau,\ d\bar{z}) - \right. \\
&\quad \left. - \frac{\bar{\tau}_2 - \bar{z}_2}{|\tau - z|^4}\frac{\partial F}{\partial \bar{z}_1}(\tau,\ d\bar{z})\right) dV_\tau.
\end{aligned}$$

In the above representation, we get the Bochner-Martinelli-like kernel for functions holomorphic in z_1 and antiholomorphic in z_2, and the volume integrals disappear for that class of functions.

Next, for $F = \begin{pmatrix} \psi_1^{11} d\bar{z}^1 & \psi_1^{12} d\bar{z}^1 \\ \psi_1^{21} d\bar{z}^1 & \psi_1^{22} d\bar{z}^1 \end{pmatrix}$ of class C^1, we have

$$\begin{aligned} 2F(z) &= -2\int_\Gamma \mathcal{U}_{(1)}(\tau,\ z)\, F(\tau,\ d\bar{z}) - \\ &\quad - 2\frac{1}{\pi^2}\int_{\Omega^+} \left(\frac{\tau_1 - z_1}{|\tau - z|^4} \frac{\partial F}{\partial z_1}(\tau,\ d\bar{z}) + \right. \\ &\quad \left. + \frac{\bar{\tau}_2 - \bar{z}_2}{|\tau - z|^4} \frac{\partial F}{\partial \bar{z}_2}(\tau,\ d\bar{z}) \right) dV_\tau, \end{aligned}$$

$$\begin{aligned} 0 &= -2\frac{1}{(2\pi i)^2}\int_\Gamma \left(\frac{\tau_2 - z_2}{|\tau - z|^4} d\bar{\tau} \wedge d\tau^2 + \right. \\ &\quad \left. + \frac{\bar{\tau}_1 - \bar{z}_1}{|\tau - z|^4} d\bar{\tau}^1 \wedge d\tau \right) F(\tau,\ d\bar{z}) - \\ &\quad - 2\frac{1}{\pi^2}\int_{\Omega^+} \left(\frac{\tau_2 - z_2}{|\tau - z|^4} \frac{\partial F}{\partial z_1}(\tau,\ d\bar{z}) - \right. \\ &\quad \left. - \frac{\bar{\tau}_1 - \bar{z}_1}{|\tau - z|^4} \frac{\partial F}{\partial \bar{z}_2}(\tau,\ d\bar{z}) \right) dV_\tau. \end{aligned}$$

What is seen here, is the "compatibility" with functions holomorphic in z_2 and antiholomorphic in z_1. Finally, for antiholomorphic (in both variables) functions we obtain

$$\begin{aligned} 2F(z) &= -2\int_\Gamma \overline{\mathcal{U}}(\tau,\ z)\, F(\tau,\ d\bar{z}) - \\ &\quad - 2\frac{1}{\pi^2}\int_{\Omega^+} \left(\frac{\tau_1 - z_1}{|\tau - z|^4} \frac{\partial F}{\partial z_1}(\tau,\ d\bar{z}) + \right. \\ &\quad \left. + \frac{\tau_2 - z_2}{|\tau - z|^4} \frac{\partial F}{\partial z_2}(\tau,\ d\bar{z}) \right) dV_\tau, \\ 0 &= -2\frac{1}{(2\pi i)^2}\int_\Gamma \left(\frac{\bar{\tau}_1 - \bar{z}_1}{|\tau - z|^4} d\bar{\tau} \wedge d\tau^1 + \right. \end{aligned}$$

$$+\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}d\bar{\tau}\wedge d\tau^2\Bigg)F(\tau,\ d\bar{z})+$$

$$+2\frac{1}{\pi^2}\int\limits_{\Omega^+}\left(\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}\frac{\partial F}{\partial z_2}(\tau,\ d\bar{z})-\right.$$

$$\left.-\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}\frac{\partial F}{\partial z_1}(\tau,\ d\bar{z})\right)dV_\tau,$$

where $F=\begin{pmatrix}\psi_{12}^{11}d\bar{z} & \psi_{12}^{12}d\bar{z}\\ \psi_{12}^{21}d\bar{z} & \psi_{12}^{22}d\bar{z}\end{pmatrix}$, $d\bar{z}=d\bar{z}_1\wedge d\bar{z}_2$.

It is quite essential to note that each one of the obtained pairs of formulas is compatible with a certain class of holomorphic-like functions in the above-mentioned meaning, i.e., the volume integrals disappear. But they may disappear not only on such classes. What is more, all of them are just particular cases, for $m=2$, of the general Borel-Pompeiu formula from Theorem 2.11, and thus are applicable to the same domain Ω. This reflects a deep idea of considering, in a fixed domain, all classes of holomorphic-like functions together, simultaneously, which will be presented in the sequel.

We will return to this subsection later when we will be able to compare it contents with the Theorem 1.7 in [HL2].

2.11.4 Notations for some integrals in $\mathbb{C}^2$

The results of these computations look quite instructive being rewritten in the operator form. For doing so let us introduce the following notation for any $F\in\mathfrak{G}_0^0(\Omega)$ or $F\in\mathcal{G}_0^0(\Omega)$ in $\mathbb{C}^2$:

$$U_1[F](z) \ := \ -\int\limits_\Gamma \mathcal{U}_1(\tau,\ z)F(\tau)\,d\bar{\tau}^2\wedge d\tau,$$

$$V_1[F](z) \ := \ \int\limits_\Gamma \mathcal{U}_1(\tau,\ z)F(\tau)\,d\bar{\tau}^1\wedge d\tau,$$

$$W_1[F](z) := -\int_\Gamma \mathcal{U}_1(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^2,$$

$$X_1[F](z) := \int_\Gamma \mathcal{U}_1(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^1,$$

$$U_2[F](z) := \int_\Gamma \mathcal{U}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau}^1 \wedge d\tau,$$

$$V_2[F](z) := -\int_\Gamma \mathcal{U}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau}^2 \wedge d\tau,$$

$$W_2[F](z) := \int_\Gamma \mathcal{U}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^1,$$

$$X_2[F](z) := -\int_\Gamma \mathcal{U}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^2,$$

$$\overline{U}_1[F](z) := -\int_\Gamma \overline{\mathcal{U}}_1(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^2,$$

$$\overline{V}_1[F](z) := \int_\Gamma \overline{\mathcal{U}}_1(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^1,$$

$$\overline{W}_1[F](z) := -\int_\Gamma \overline{\mathcal{U}}_1(\tau,\ z)\, F(\tau)\, d\bar{\tau}^2 \wedge d\tau,$$

$$\overline{X}_1[F](z) := \int_\Gamma \overline{\mathcal{U}}_1(\tau,\ z)\, F(\tau)\, d\bar{\tau}^1 \wedge d\tau,$$

$$\overline{U}_2[F](z) := \int_\Gamma \overline{\mathcal{U}}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^1,$$

$$\overline{V}_2[F](z) := -\int_\Gamma \overline{\mathcal{U}}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge d\tau^2,$$

$$\overline{W}_2[F](z) := \int_\Gamma \overline{\mathcal{U}}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau}^1 \wedge d\tau,$$

$$\overline{X}_2[F](z) := -\int_\Gamma \overline{\mathcal{U}}_2(\tau,\ z)\, F(\tau)\, d\bar{\tau}^2 \wedge d\tau,$$

$$T_1[F](z) := -(2i)^2 \int_{\Omega^+} \mathcal{U}_1(\tau,\ z)\, F(\tau)\, dV_\tau,$$

$$T_2[F](z) := -(2i)^2 \int_{\Omega^+} \mathcal{U}_2(\tau,\ z)\, F(\tau)\, dV_\tau,$$

$$\overline{T}_1[F](z) := -(2i)^2 \int_{\Omega^+} \overline{\mathcal{U}}_1(\tau,\ z)\, F(\tau)\, dV_\tau,$$

$$\overline{T}_2[F](z) := -(2i)^2 \int_{\Omega^+} \overline{\mathcal{U}}_2(\tau,\ z)\, F(\tau)\, dV_\tau,$$

and

$$U[F](z) := -\int_{\Gamma} \mathcal{U}(\tau,\ z)\, F(\tau)\,,$$

$$U_{(2)}[F](z) := -\int_{\Gamma} \mathcal{U}_{(2)}(\tau,\ z)\, F(\tau)\,,$$

$$U_{(1)}[F](z) := -\int_{\Gamma} \mathcal{U}_{(1)}(\tau,\ z)\, F(\tau)\,,$$

$$\overline{U}[F](z) := -\int_{\Gamma} \overline{\mathcal{U}}(\tau,\ z)\, F(\tau)\,.$$

Now we resume the above computations in the following theorems.

2.11.5 Formulas of the Borel-Pompeiu type in $\mathbb{C}^2$

Theorem *Let $\Omega^+ \subset \mathbb{C}^2$ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $f \in \overline{\mathcal{G}}_1^0(\Omega^+) \cap \overline{\mathcal{G}}_0^0(\Omega^+ \cup \Gamma)$. Then the following equalities hold in Ω^+:*

$$f = U[f] + T_1\left[\frac{\partial f}{\partial \bar{z}_1}\right] + T_2\left[\frac{\partial f}{\partial \bar{z}_2}\right],$$

$$f = U_{(2)}[f] + T_1\left[\frac{\partial f}{\partial \bar{z}_1}\right] + \overline{T}_2\left[\frac{\partial f}{\partial z_2}\right],$$

$$f = U_{(1)}[f] + \overline{T}_1\left[\frac{\partial f}{\partial z_1}\right] + T_2\left[\frac{\partial f}{\partial \bar{z}_2}\right],$$

$$f = \overline{U}[f] + \overline{T}_1\left[\frac{\partial f}{\partial z_1}\right] + \overline{T}_2\left[\frac{\partial f}{\partial z_2}\right],$$

which are particular cases, or fragments, of the combined formula (2.11.6) in $\mathbb{C}^2$ (see (2.11.1) also).

2.11.6 Complements to the Borel-Pompeiu-type formulas in $\mathbb{C}^2$

Theorem. *Let $\Omega^+ \subset \mathbb{C}^2$ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $f \in \overline{\mathcal{G}}_1^0(\Omega^+) \cap \overline{\mathcal{G}}_0^0(\Omega^+ \cup \Gamma)$. Then the following equalities hold in Ω^+:*

$$0 = \overline{V}_2[f] - V_1[f] + \overline{T}_2\left[\frac{\partial f}{\partial z_1}\right] - T_1\left[\frac{\partial f}{\partial \bar{z}_2}\right],$$

$$0 = X_1[f] - X_2[f] + T_1\left[\frac{\partial f}{\partial z_2}\right] - T_2\left[\frac{\partial f}{\partial z_1}\right],$$

which are, again, particular cases, or fragments of the combined formula (2.11.7) in $\mathbb{C}^2$ (see (2.11.1) also).

2.11.7 The case $m > 2$

The case of $m = 2$, studied in detail, is a very good model of the general situation of an arbitrary $m \geq 2$, which allows us not to repeat all the reasonings and just to write down the conclusions. To this end we assume that

$$F = \begin{pmatrix} \psi_{\mathbf{j}}^{11} d\bar{z}^{\mathbf{j}} & \psi_{\mathbf{j}}^{12} d\bar{z}^{\mathbf{j}} \\ \psi_{\mathbf{j}}^{21} d\bar{z}^{\mathbf{j}} & \psi_{\mathbf{j}}^{22} d\bar{z}^{\mathbf{j}} \end{pmatrix} = F_{\mathbf{j}} d\bar{z}^{\mathbf{j}},$$

with $F_{\mathbf{j}} = \begin{pmatrix} \psi_{\mathbf{j}}^{11} & \psi_{\mathbf{j}}^{12} \\ \psi_{\mathbf{j}}^{21} & \psi_{\mathbf{j}}^{22} \end{pmatrix}$ being of class C^1, then we have

$$\begin{aligned} 2F_{\mathbf{j}}(z) &= (-1)^{\frac{m(m-1)}{2}} 2 \int_\Gamma \mathcal{U}_{\mathbf{j}}(\tau, z) F_{\mathbf{j}}(\tau) - \\ &- 2\frac{(m-1)!}{\pi^m} \int_{\Omega^+} \left\{ \sum_{\substack{j=1 \\ j \neq j_1, \ldots, j_{|\mathbf{j}|}}}^{m} \frac{\bar{\tau}_j - \bar{z}_j}{|\tau - z|^{2m}} \frac{\partial F_{\mathbf{j}}}{\partial \bar{z}_j}(\tau) + \right. \\ &\left. + \sum_{j=j_1, \ldots, j_{|\mathbf{j}|}} \frac{\tau_j - z_j}{|\tau - z|^{2m}} \frac{\partial F_{\mathbf{j}}}{\partial z_j}(\tau) \right\} dV_\tau. \end{aligned}$$

Also, for $|\mathbf{j}| = 1,\ \ldots,\ m-1$, we have

$$\begin{aligned} 0 \;=\; & (-1)^{\frac{m(m-1)}{2}} (-1)^{p+q} (-1)^{\frac{(p-q)-|p-q|}{2(p-q)}} 2 \frac{(m-1)!}{(2\pi i)^m} \cdot \\ & \cdot \left\{ \int_\Gamma (-1)^{k_q - 1} \frac{\bar{\tau}_{j_p} - \bar{z}_{j_p}}{|\tau - z|^{2m}} d\bar{\tau}_{[k_q]} \wedge d\tau F_{\mathbf{j}}(\tau) - \right. \\ & \left. - (-1)^m (-1)^{j_p - 1} \int_\Gamma \frac{\tau_{k_q} - z_{k_q}}{|\tau - z|^{2m}} d\bar{\tau} \wedge d\tau_{[j_p]} F_{\mathbf{j}}(\tau) \right\} - \\ & - (-1)^{p+q} (-1)^{\frac{(p-q)-|p-q|}{2(p-q)}} 2 \frac{(m-1)!}{\pi^m} \cdot \\ & \cdot \left\{ \int_{\Omega^+} \frac{\bar{\tau}_{j_p} - \bar{z}_{j_p}}{|\tau - z|^{2m}} \frac{\partial F_{\mathbf{j}}}{\partial \bar{z}_{k_q}}(\tau)\, dV_\tau - \int_{\Omega^+} \frac{\tau_{k_q} - z_{k_q}}{|\tau - z|^{2m}} \frac{\partial F_{\mathbf{j}}}{\partial z_{j_p}}(\tau)\, dV_\tau \right\}, \end{aligned}$$

where $k_q = 1,\ \ldots,\ m$ with $j_q < k_q < j_{q+1}$ for any $q = 0,\ \ldots,\ |\mathbf{j}|$, by definition $j_0 := 0$ and $j_{|\mathbf{j}|+1} := m+1$ and for $p = q$, $\frac{(p-q)-|p-q|}{2(p-q)} := 1$.

For $|\mathbf{j}| = 0,\ \ldots,\ m-2$, we have

$$\begin{aligned} 0 \;=\; & (-1)^{\frac{m(m-1)}{2}} 2 \int_\Gamma \Big((-1)^{j-1} \overline{\mathcal{U}}_q(\tau,\ z)\, d\bar{\tau}_{[j]} \wedge d\tau - \\ & - (-1)^{q-1} \overline{\mathcal{U}}_j(\tau,\ z)\, d\bar{\tau}_{[q]} \wedge d\tau \Big) F_{\mathbf{j}}(\tau) - \\ & - 2 (2i)^m \int_{\Omega^+} \left(\overline{\mathcal{U}}_q(\tau,\ z) \frac{\partial F_{\mathbf{j}}}{\partial \bar{z}_j}(\tau) - \overline{\mathcal{U}}_j(\tau,\ z) \frac{\partial F_{\mathbf{j}}}{\partial \bar{z}_q}(\tau) \right) dV_\tau, \end{aligned}$$

where $j,\ q = 1,\ \ldots,\ m$, $j,\ q \neq j_1,\ \ldots,\ j_{|\mathbf{j}|}$ and $q < j$.

Finally, for $|\mathbf{j}| = 2,\ \ldots,\ m$, we have

$$\begin{aligned} 0 \;=\; & (-1)^{\frac{m(m-1)}{2}} 2 (-1)^m \int_\Gamma \Big((-1)^{j_q - 1} \mathcal{U}_{j_p}(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j_q]} - \\ & - (-1)^{j_p - 1} \mathcal{U}_{j_q}(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j_p]} \Big) F_{\mathbf{j}}(\tau) - \\ & - 2 (2i)^m \int_{\Omega^+} \left(\mathcal{U}_{j_p}(\tau,\ z) \frac{\partial F_{\mathbf{j}}}{\partial z_{j_q}}(\tau) - \mathcal{U}_{j_q}(\tau,\ z) \frac{\partial F_{\mathbf{j}}}{\partial z_{j_p}}(\tau) \right) dV_\tau, \end{aligned}$$

where $p,\ q = 1,\ \ldots,\ |\mathbf{j}|$ and $p < q$.

Because all participating operators are linear, the case of an arbitrary form does not give any new formula, only the same as that considered above.

2.11.8 Notations for some integrals in $\mathbb{C}^m$

We are going to present again the formulas from the previous Subsection 2.11.7 in the operator form, thus, generalizing the notations from Subsection 2.11.4, let us introduce the following notation for any $F \in \mathfrak{G}_0^0(\Omega)$ or $F \in \mathcal{G}_0^0(\Omega)$:

$$U_{jk}[F](z) := (-1)^{\frac{m(m-1)}{2}} \int\limits_{\Gamma} (-1)^{k-1} \mathcal{U}_j(\tau,\ z)\, F(\tau)\, d\bar{\tau}_{[k]} \wedge d\tau,$$

$$W_{jk}[F](z) := (-1)^{\frac{m(m-1)}{2}} (-1)^m \int\limits_{\Gamma} (-1)^{k-1} \mathcal{U}_j(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge \wedge d\tau_{[k]},$$

$$\overline{U}_{jk}[F](z) := (-1)^{\frac{m(m-1)}{2}} (-1)^m \int\limits_{\Gamma} (-1)^{k-1} \overline{\mathcal{U}}_j(\tau,\ z)\, F(\tau)\, d\bar{\tau} \wedge \wedge d\tau_{[k]},$$

$$\overline{W}_{jk}[F](z) := (-1)^{\frac{m(m-1)}{2}} \int\limits_{\Gamma} (-1)^{k-1} \overline{\mathcal{U}}_j(\tau,\ z)\, F(\tau)\, d\bar{\tau}_{[k]} \wedge d\tau,$$

$$\begin{aligned} T_j[F](z) &:= -(2i)^m \int\limits_{\Omega^+} \mathcal{U}_j(\tau,\ z)\, F(\tau)\, dV_\tau, \\ \overline{T}_j[F](z) &:= -(2i)^m \int\limits_{\Omega^+} \overline{\mathcal{U}}_j(\tau,\ z)\, F(\tau)\, dV_\tau, \end{aligned}$$

where $j,\ k = 1,\ \ldots,\ m$, and

$$\begin{aligned} U_{\mathbf{j}}[F](z) &:= (-1)^{\frac{m(m-1)}{2}} \int\limits_{\Gamma} \mathcal{U}_{\mathbf{j}}(\tau,\ z)\, F(\tau), \\ U[F](z) &:= U_{\emptyset}[F] = (-1)^{\frac{m(m-1)}{2}} \int\limits_{\Gamma} \mathcal{U}(\tau,\ z)\, F(\tau), \end{aligned}$$

$$\overline{U}\left[F\right](z) \quad := \quad U_{1\ldots m}\left[F\right] = (-1)^{\frac{m(m-1)}{2}} \int\limits_{\Gamma} \overline{\mathcal{U}}\left(\tau,\ z\right) F\left(\tau\right),$$

where $\mathbf{j}$ is a strictly increasing $|\mathbf{j}|$-tuple in $\{1,\ \ldots,\ m\}$.

Now we resume the above computations in the following theorems.

2.11.9 Formulas of the Borel-Pompeiu type in $\mathbb{C}^m$

Theorem *Let $\Omega^+ \subset \mathbb{C}^m$ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $f \in \overline{\mathcal{G}}_1^0\left(\Omega^+\right) \cap \overline{\mathcal{G}}_0^0\left(\Omega^+ \cup \Gamma\right)$. Then the following equalities hold in Ω^+:*

$$f = U_{\mathbf{j}}\left[f\right] + \sum_{\substack{j=1 \\ j \neq j_1,\ \ldots,\ j_{|\mathbf{j}|}}}^{m} T_j\left[\frac{\partial f}{\partial \bar{z}_j}\right] + \sum_{j=j_1,\ldots,\ j_{|\mathbf{j}|}} \overline{T}_j\left[\frac{\partial f}{\partial z_j}\right],$$

where $\mathbf{j}$ is a strictly increasing $|\mathbf{j}|$-tuple in $\{1,\ \ldots,\ m\}$, specifically,

$$\begin{aligned} f &= U\left[f\right] + \sum_{j=1}^{m} T_j\left[\frac{\partial f}{\partial \bar{z}_j}\right], \\ f &= \overline{U}\left[f\right] + \sum_{j=1}^{m} \overline{T}_j\left[\frac{\partial f}{\partial z_j}\right]. \end{aligned}$$

They are particular cases, or fragments, of the combined formula (2.11.6), see also (2.11.1).

2.11.10 Complements to the Borel-Pompeiu-type formulas in $\mathbb{C}^m$

Theorem. *Let $\Omega^+ \subset \mathbb{C}^m$ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $f \in \overline{\mathcal{G}}_1^0\left(\Omega^+\right) \cap \overline{\mathcal{G}}_0^0\left(\Omega^+ \cup \Gamma\right)$. Then the following equalities hold in Ω^+ for any $j \neq k$, $j,\ k = 1,\ \ldots,\ m$:*

$$0 \quad = \quad U_{jk}\left[f\right] - \overline{U}_{kj}\left[f\right] + T_j\left[\frac{\partial f}{\partial \bar{z}_k}\right] - \overline{T}_k\left[\frac{\partial f}{\partial z_j}\right],$$

$$
\begin{aligned}
0 &= \overline{W}_{jk}[f] - \overline{W}_{kj}[f] + \overline{T}_j \left[\frac{\partial f}{\partial \bar{z}_k}\right] - \overline{T}_k \left[\frac{\partial f}{\partial \bar{z}_j}\right], \\
0 &= W_{jk}[f] - W_{kj}[f] + T_j \left[\frac{\partial f}{\partial z_k}\right] - T_k \left[\frac{\partial f}{\partial z_j}\right].
\end{aligned}
$$

which are particular cases, or fragments, of the combined formula (2.11.7), see also (2.11.1).

Chapter 3

Hyperholomorphic functions and differential forms in $\mathbb{C}^m$

3.1 Hyperholomorphy in $\mathbb{C}^m$

Definition *We set*

$$\overline{\mathfrak{N}}(\Omega) \ := \ \ker\left(\overline{\mathcal{D}}\right),$$

and we shall call m.v.d.f. from $\overline{\mathfrak{N}}(\Omega)$ *(left)-hyperholomorphic in*Ω *in the sense of* $\overline{\mathcal{D}}$*. Analogously we set*

$$\mathfrak{N}(\Omega) \ := \ \ker(\mathcal{D}),$$

and we shall call the corresponding elements (left)-hyperholomorphic in the sense of $\mathcal{D}$*.*

Formally, one can introduce two more classes of **m.v.d.f.** analogous to the above: $\overline{\mathfrak{N}}^*(\Omega) \ := \ \ker\left(\overline{\mathcal{D}}^*\right)$, $\overline{\mathfrak{N}}^*(\Omega) \ := \ \ker\left(\mathcal{D}^*\right)$. But it follows directly from (2.3.8) that $\left(\ker\overline{\mathcal{D}}^*\right) = \ker\overline{\mathcal{D}}$ and $\ker\mathcal{D}^* = \ker\mathcal{D}$. Thus we shall deal, as a rule, with the set $\overline{\mathfrak{N}}(\Omega)$ only.

The right-hand-side Cauchy-Riemann operators cannot be used thus directly, and we introduce the following class of hyperholomorphy.

For a fixed **m.v.d.f.** G of class C^0 we define the set

$$\overline{\mathfrak{M}}_{r;\,G}\left(\Omega\right) \ := \ \left\{F \in \overline{\mathfrak{G}}_1\left(\Omega\right) | \overline{\mathcal{D}}_r\left[F\right] \underset{\star}{\wedge} G = 0\right\},$$

and its elements are called *right-hyperholomorphic in the sense of* $\overline{\mathcal{D}}_r$ *and with respect to* G. Analogously for the set

$$\mathfrak{M}_{r;\,G}(\Omega) := \left\{ F \in \mathfrak{G}_1(\Omega) \,|\, \mathcal{D}_r[F] \underset{\star}{\wedge} G = 0 \right\}.$$

Normally we shall be working with the operator $\overline{\mathcal{D}}$ referring to the corresponding elements as just hyperholomorphic if no misunderstanding can arise. For example, it then follows from (2.3.3) and (2.3.4) that $F := \left(F^{\alpha\beta}\right)_{1\le\alpha\le 2}^{1\le\beta\le 2}$ is in $\overline{\mathfrak{N}}(\Omega)$ iff in Ω

$$\begin{array}{ll} \bar\partial \wedge F^{11} + \bar\partial^* \wedge F^{21} &= 0, \\ \bar\partial^* \wedge F^{11} + \bar\partial \wedge F^{21} &= 0, \\ \bar\partial \wedge F^{12} + \bar\partial^* \wedge F^{22} &= 0, \\ \bar\partial^* \wedge F^{12} + \bar\partial \wedge F^{22} &= 0. \end{array} \tag{3.1.1}$$

Of course, these equalities can be considered as *the Cauchy-Riemann conditions for the matrix* F *to be left-hyperholomorphic*. Below we present more detailed analysis of (3.1.1).

3.2 Hyperholomorphy in one variable

Let here $m = 1$, i.e., we are interested in the above-introduced classes in $\mathbb{C} = \mathbb{C}^1$. Let Ω be a domain in $\mathbb{C}$ and $F \in \overline{\mathfrak{G}}_1(\Omega)$, so that any $F^{\alpha\beta}$ is of the form $\varphi^{\alpha\beta} + \psi^{\alpha\beta} d\bar z^1$ for $\alpha, \beta \in \{1,\ 2\}$ where $\varphi^{\alpha\beta}$ and $\psi^{\alpha\beta}$ are from $C^1(\Omega;\ \mathbb{C})$, and let them satisfy the system (3.1.1). Hence we have, for φ^{11}, ψ^{11}, φ^{21}, ψ^{21},

$$\begin{cases} \frac{\partial\varphi^{11}}{\partial\bar z_1} d\bar z^1 + \frac{\partial\psi^{11}}{\partial\bar z_1} d\bar z^1 \wedge d\bar z^1 + \bar\partial^*\left[\varphi^{21}\right] + \frac{\partial\psi^{21}}{\partial z_1}\widehat{d\bar z^1} \wedge d\bar z^1 &= 0, \\ \bar\partial^*\left[\varphi^{11}\right] + \frac{\partial\psi^{11}}{\partial z_1}\widehat{d\bar z^1} \wedge d\bar z^1 + \frac{\partial\varphi^{21}}{\partial\bar z_1} d\bar z^1 + \frac{\partial\psi^{21}}{\partial\bar z_1} d\bar z^1 \wedge d\bar z^1 &= 0, \end{cases}$$

or equivalently,

$$\begin{cases} \frac{\partial\varphi^{11}}{\partial\bar z_1} d\bar z^1 + \frac{\partial\psi^{21}}{\partial z_1} &= 0, \\ \frac{\partial\psi^{11}}{\partial z_1} + \frac{\partial\varphi^{21}}{\partial\bar z_1} d\bar z^1 &= 0, \end{cases}$$

which means that the functions φ^{11} and φ^{21} are holomorphic, while ψ^{11} and ψ^{21} are antiholomorphic in Ω. Analogously for φ^{12}, ψ^{12}, φ^{22}, ψ^{22}. Concluding, we can write

$$\overline{\mathfrak{N}}(\Omega) = \left\{ \left(\varphi^{\alpha\beta} + \psi^{\alpha\beta} d\bar{z}^1 \right)_{1\leq\alpha\leq 2}^{1\leq\beta\leq 2} \right\},$$

with $\left\{\varphi^{\alpha\beta}\right\} \subset Hol(\Omega;\, \mathbb{C})$, $\left\{\psi^{\alpha\beta}\right\} \subset \overline{Hol}(\Omega;\, \mathbb{C})$, where Hol concerns holomorphic, and $\overline{Hol}$ antiholomorphic functions.

Observe that for any 0-matrix-form F and for any 1-matrix-form G from $\overline{\mathfrak{N}}(\Omega)$ there holds $\bar{\partial}^*[F] = 0$ and $\bar{\partial}[G] = 0$ in Ω, that is, all such differential forms are, respectively, $\bar{\partial}^*$-closed and $\bar{\partial}$-closed.

One more observation. We see that the set $\overline{\mathfrak{N}}(\Omega)$ contains simultaneously both holomorphic and antiholomorphic functions in the same domain, but the latter are identified with the coefficients of specific differential forms.

Only this way allows us to consider holomorphic and antiholomorphic functions (in the same domain) as elements of a single set which possesses a deep structural analogy with the set of only holomorphic functions (and, separately, with that of only antiholomorphic ones).

One more justification of the necessity to combine both classes into a single one is that the case of more than one independent variable requires exactly this approach, which will be shown in what follows.

Of course, it cannot be achieved without loss: $\overline{\mathfrak{N}}(\Omega)$ is **not** an algebra with respect to the wedge-multiplication.

3.3 Hyperholomorphy in two variables

Now we take $m = 2$, that is,

$$F^{\alpha\beta} = \varphi^{\alpha\beta} + \sum_{\gamma=1}^{2} \psi_{\gamma}^{\alpha\beta} d\bar{z}^{\gamma} + \psi_{12}^{\alpha\beta} d\bar{z}^1 \wedge d\bar{z}^2.$$

Straightforward computation gives

$$\bar{\partial}\left[F^{\alpha\beta}\right] = \frac{\partial\varphi^{\alpha\beta}}{\partial\bar{z}_1} d\bar{z}^1 + \frac{\partial\varphi^{\alpha\beta}}{\partial\bar{z}_2} d\bar{z}^2 + \left(\frac{\partial\psi_2^{\alpha\beta}}{\partial\bar{z}_1} - \frac{\partial\psi_1^{\alpha\beta}}{\partial\bar{z}_2} \right) d\bar{z}^1 \wedge d\bar{z}^2,$$

$$\bar{\partial}^* \left[F^{\alpha\beta}\right] = \frac{\partial \psi_1^{\alpha\beta}}{\partial z_1} + \frac{\partial \psi_2^{\alpha\beta}}{\partial z_2} - \frac{\partial \psi_{12}^{\alpha\beta}}{\partial z_2} d\bar{z}^1 + \frac{\partial \psi_{12}^{\alpha\beta}}{\partial z_1} d\bar{z}^2.$$

Hence, the first two equations in (3.1.1) take the form

$$\begin{cases} \left(\frac{\partial \psi_1^{21}}{\partial z_1} + \frac{\partial \psi_2^{21}}{\partial z_2}\right) + \left(\frac{\partial \varphi^{11}}{\partial \bar{z}_1} - \frac{\partial \psi_{12}^{21}}{\partial z_2}\right) d\bar{z}^1 + \\ + \left(\frac{\partial \varphi^{11}}{\partial \bar{z}_2} + \frac{\partial \psi_{12}^{21}}{\partial z_1}\right) d\bar{z}^2 + \left(\frac{\partial \psi_2^{11}}{\partial \bar{z}_1} - \frac{\partial \psi_1^{11}}{\partial \bar{z}_2}\right) d\bar{z}^1 \wedge d\bar{z}^2 = 0, \\ \left(\frac{\partial \psi_1^{11}}{\partial z_1} + \frac{\partial \psi_2^{11}}{\partial z_2}\right) - \left(\frac{\partial \psi_{12}^{11}}{\partial z_2} - \frac{\partial \varphi^{21}}{\partial \bar{z}_1}\right) d\bar{z}^1 + \\ + \left(\frac{\partial \psi_{12}^{11}}{\partial z_1} + \frac{\partial \varphi^{21}}{\partial \bar{z}_2}\right) d\bar{z}^2 - \left(\frac{\partial \psi_1^{21}}{\partial \bar{z}_2} - \frac{\partial \psi_2^{21}}{\partial \bar{z}_1}\right) d\bar{z}^1 \wedge d\bar{z}^2 = 0. \end{cases}$$

The same for F^{12} and F^{22}. Thus $F = \left(F^{\alpha\beta}\right)_{1\leq\alpha\leq 2}^{1\leq\beta\leq 2}$ belongs to $\overline{\mathfrak{N}}(\Omega)$ if and only if the following hyperholomorphic Cauchy-Riemann conditions hold:

$$\begin{cases} \frac{\partial \psi_1^{21}}{\partial z_1} + \frac{\partial \psi_2^{21}}{\partial z_2} = 0, & \frac{\partial \psi_1^{11}}{\partial z_1} + \frac{\partial \psi_2^{11}}{\partial z_2} = 0, \\ \frac{\partial \varphi^{11}}{\partial \bar{z}_1} - \frac{\partial \psi_{12}^{21}}{\partial z_2} = 0, & \frac{\partial \varphi^{21}}{\partial \bar{z}_1} - \frac{\partial \psi_{12}^{11}}{\partial z_2} = 0, \\ \frac{\partial \varphi^{11}}{\partial \bar{z}_2} + \frac{\partial \psi_{12}^{21}}{\partial z_1} = 0, & \frac{\partial \psi_{12}^{11}}{\partial z_1} + \frac{\partial \varphi^{21}}{\partial \bar{z}_2} = 0, \\ \frac{\partial \psi_2^{11}}{\partial \bar{z}_1} - \frac{\partial \psi_1^{11}}{\partial \bar{z}_2} = 0, & \frac{\partial \psi_2^{21}}{\partial \bar{z}_1} - \frac{\partial \psi_1^{21}}{\partial \bar{z}_2} = 0, \end{cases} \tag{3.3.1}$$

$$\begin{cases} \frac{\partial \psi_1^{22}}{\partial z_1} + \frac{\partial \psi_2^{22}}{\partial z_2} = 0, & \frac{\partial \psi_1^{12}}{\partial z_1} + \frac{\partial \psi_2^{12}}{\partial z_2} = 0, \\ \frac{\partial \varphi^{12}}{\partial \bar{z}_1} - \frac{\partial \psi_{12}^{22}}{\partial z_2} = 0, & \frac{\partial \varphi^{22}}{\partial \bar{z}_1} - \frac{\partial \psi_{12}^{12}}{\partial z_2} = 0, \\ \frac{\partial \varphi^{12}}{\partial \bar{z}_2} + \frac{\partial \psi_{12}^{22}}{\partial z_1} = 0, & \frac{\partial \psi_{12}^{12}}{\partial z_1} + \frac{\partial \varphi^{22}}{\partial \bar{z}_2} = 0, \\ \frac{\partial \psi_2^{12}}{\partial \bar{z}_1} - \frac{\partial \psi_1^{12}}{\partial \bar{z}_2} = 0, & \frac{\partial \psi_2^{22}}{\partial \bar{z}_1} - \frac{\partial \psi_1^{22}}{\partial \bar{z}_2} = 0. \end{cases} \tag{3.3.2}$$

In particular they mean that φ^{11}, φ^{21}, φ^{12} and φ^{22} can be taken holomorphic in two variables and ψ_{12}^{21}, ψ_{12}^{11}, ψ_{12}^{22} and ψ_{12}^{12} are taken

antiholomorphic while ψ_1^{21}, ψ_1^{11}, ψ_1^{22} and ψ_1^{12} are taken antiholomorphic in the variable z_1 and holomorphic in the variable z_2 and ψ_2^{21}, ψ_2^{11}, ψ_2^{22} and ψ_2^{12} are taken holomorphic in the variable z_1 and antiholomorphic in the variable z_2. In general, $\overline{\mathfrak{N}}(\Omega)$ contains, as proper subspaces:

1. the set $Hol\left(\Omega;\ \mathbb{C}^2\right)$ of all holomorphic mappings,
2. the set isomorphic to the set $\overline{Hol}\left(\Omega;\mathbb{C}^2\right)$ of all antiholomorphic mappings, with their coordinate functions being identified with the coefficients of specific differential forms,
3. the sets isomorphic with the sets of mappings, whose coordinate functions are holomorphic with respect to some variable and antiholomorphic with respect to the other, where again it is necessary to identify coordinate functions with certain differential forms.

But $\overline{\mathfrak{N}}(\Omega)$ is not exhausted with them; for example, the matrices

$$\left(\begin{array}{cc} e^{2(Re(z_2)+iIm(z_1))}; & 0 \\ -e^{2(Re(z_2)+iIm(z_1))}d\bar{z}^1\wedge d\bar{z}^2; & 0 \end{array}\right),$$

$$\left(\begin{array}{cc} 0; & 0 \\ 0; & e^{2(Re(z_1)-iIm(z_2))}\left(d\bar{z}^1+d\bar{z}^2\right) \end{array}\right)$$

are in $\overline{\mathfrak{N}}(\Omega)$, but their non-zero entries do not belong to any of the above described sets, i.e., their coefficients are neither holomorphic nor antiholomorphic with respect to each one of the variables z_1 and z_2.

3.4 Hyperholomorphy in three variables

Now let $m=3$. Then the elements of the matrices are of the form

$$F^{\alpha\beta}=\varphi^{\alpha\beta}+\sum_{k=1}^{3}\sum_{|\gamma|=k}\psi_{\gamma}^{\alpha\beta}d\bar{z}^{\gamma}.$$

Again straightforwardly one obtains

$$\bar{\partial}\left[F^{\alpha\beta}\right] \ = \ \sum_{q=1}^{3}\frac{\partial\varphi^{\alpha\beta}}{\partial\bar{z}_q}d\bar{z}^q+\left(\frac{\partial\psi_3^{\alpha\beta}}{\partial\bar{z}_2}-\frac{\partial\psi_2^{\alpha\beta}}{\partial\bar{z}_3}\right)d\bar{z}^2\wedge d\bar{z}^3+$$

$$+\left(\frac{\partial\psi_3^{\alpha\beta}}{\partial\bar{z}_1}-\frac{\partial\psi_1^{\alpha\beta}}{\partial\bar{z}_3}\right)d\bar{z}^1\wedge d\bar{z}^3+$$
$$+\left(\frac{\partial\psi_2^{\alpha\beta}}{\partial\bar{z}_1}-\frac{\partial\psi_1^{\alpha\beta}}{\partial\bar{z}_2}\right)d\bar{z}^1\wedge d\bar{z}^2+$$
$$\left(\frac{\partial\psi_{23}^{\alpha\beta}}{\partial\bar{z}_1}-\frac{\partial\psi_{13}^{\alpha\beta}}{\partial\bar{z}_2}+\frac{\partial\psi_{12}^{\alpha\beta}}{\partial\bar{z}_3}\right)d\bar{z}^1\wedge d\bar{z}^2\wedge d\bar{z}^3,$$

$$\begin{aligned}\bar{\partial}^*\left[F^{\alpha\beta}\right] &= \left(\frac{\partial\psi_1^{\alpha\beta}}{\partial z_1}+\frac{\partial\psi_2^{\alpha\beta}}{\partial z_2}+\frac{\partial\psi_3^{\alpha\beta}}{\partial z_3}\right)-\left(\frac{\partial\psi_{13}^{\alpha\beta}}{\partial z_3}+\frac{\partial\psi_{12}^{\alpha\beta}}{\partial z_2}\right)d\bar{z}^1+\\ &+\left(\frac{\partial\psi_{12}^{\alpha\beta}}{\partial z_1}-\frac{\partial\psi_{23}^{\alpha\beta}}{\partial z_3}\right)d\bar{z}^2+\left(\frac{\partial\psi_{23}^{\alpha\beta}}{\partial z_2}+\frac{\partial\psi_{13}^{\alpha\beta}}{\partial z_1}\right)d\bar{z}^3+\\ &\frac{\partial\psi_{123}^{\alpha\beta}}{\partial z_1}d\bar{z}^2\wedge d\bar{z}^3-\frac{\partial\psi_{123}^{\alpha\beta}}{\partial z_2}d\bar{z}^1\wedge d\bar{z}^3+\frac{\partial\psi_{123}^{\alpha\beta}}{\partial z_3}d\bar{z}^1\wedge d\bar{z}^2.\end{aligned}$$

Thus, the first two equations in (3.1.1) take the form

$$
\begin{cases}
\left(\frac{\partial \psi_1^{21}}{\partial z_1}+\frac{\partial \psi_2^{21}}{\partial z_2}+\frac{\partial \psi_3^{21}}{\partial z_3}\right)+\left(\frac{\partial \varphi^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{21}}{\partial z_3}-\frac{\partial \psi_{12}^{21}}{\partial z_2}\right) d\bar{z}^1+ \\
\qquad +\left(\frac{\partial \varphi^{11}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{21}}{\partial z_1}-\frac{\partial \psi_{23}^{21}}{\partial z_3}\right) d\bar{z}^2+ \\
\qquad +\left(\frac{\partial \varphi^{11}}{\partial \bar{z}_3}+\frac{\partial \psi_{23}^{21}}{\partial z_2}+\frac{\partial \psi_{13}^{21}}{\partial z_1}\right) d\bar{z}^3+ \\
\quad +\left(\frac{\partial \psi_3^{11}}{\partial \bar{z}_2}-\frac{\partial \psi_2^{11}}{\partial \bar{z}_3}+\frac{\partial \psi_{123}^{21}}{\partial z_1}\right) d\bar{z}^2\wedge d\bar{z}^3+ \\
\quad +\left(\frac{\partial \psi_3^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{11}}{\partial \bar{z}_3}-\frac{\partial \psi_{123}^{21}}{\partial z_2}\right) d\bar{z}^1\wedge d\bar{z}^3+ \\
\quad +\left(\frac{\partial \psi_2^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{11}}{\partial \bar{z}_2}+\frac{\partial \psi_{123}^{21}}{\partial z_3}\right) d\bar{z}^1\wedge d\bar{z}^2+ \\
\quad +\left(\frac{\partial \psi_{23}^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{11}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{11}}{\partial \bar{z}_3}\right) d\bar{z}^1\wedge d\bar{z}^2\wedge d\bar{z}^3 \quad =0, \\
\left(\frac{\partial \psi_1^{11}}{\partial z_1}+\frac{\partial \psi_2^{11}}{\partial z_2}+\frac{\partial \psi_3^{11}}{\partial z_3}\right)+\left(\frac{\partial \varphi^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{11}}{\partial z_3}-\frac{\partial \psi_{12}^{11}}{\partial z_2}\right) d\bar{z}^1+ \\
\qquad +\left(\frac{\partial \varphi^{21}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{11}}{\partial z_1}-\frac{\partial \psi_{23}^{11}}{\partial z_3}\right) d\bar{z}^2+ \\
\qquad +\left(\frac{\partial \varphi^{21}}{\partial \bar{z}_3}+\frac{\partial \psi_{23}^{11}}{\partial z_2}+\frac{\partial \psi_{13}^{11}}{\partial z_1}\right) d\bar{z}^3+ \\
\quad +\left(\frac{\partial \psi_3^{21}}{\partial \bar{z}_2}-\frac{\partial \psi_2^{21}}{\partial \bar{z}_3}+\frac{\partial \psi_{123}^{11}}{\partial z_1}\right) d\bar{z}^2\wedge d\bar{z}^3+ \\
\quad +\left(\frac{\partial \psi_3^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{21}}{\partial \bar{z}_3}-\frac{\partial \psi_{123}^{11}}{\partial z_2}\right) d\bar{z}^1\wedge d\bar{z}^3+ \\
\quad +\left(\frac{\partial \psi_2^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{21}}{\partial \bar{z}_2}+\frac{\partial \psi_{123}^{11}}{\partial z_3}\right) d\bar{z}^1\wedge d\bar{z}^2+ \\
\quad +\left(\frac{\partial \psi_{23}^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{21}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{21}}{\partial \bar{z}_3}\right) d\bar{z}^1\wedge d\bar{z}^2\wedge d\bar{z}^3 \quad =0.
\end{cases}
$$

The same for F^{12} and F^{22}.

Thus $F=\left(F^{\alpha\beta}\right)_{1\leq\alpha\leq 2}^{1\leq\beta\leq 2}$ belongs to $\overline{\mathfrak{N}}(\Omega)$ in $\mathbb{C}^3$ if and only if the following hyperholomorphic Cauchy-Riemann conditions hold:

$$
\begin{cases}
\frac{\partial \psi_1^{21}}{\partial z_1}+\frac{\partial \psi_2^{21}}{\partial z_2}+\frac{\partial \psi_3^{21}}{\partial z_3} = 0, & \frac{\partial \psi_1^{11}}{\partial z_1}+\frac{\partial \psi_2^{11}}{\partial z_2}+\frac{\partial \psi_3^{11}}{\partial z_3} = 0,\\
\frac{\partial \varphi^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{21}}{\partial z_3}-\frac{\partial \psi_{12}^{21}}{\partial z_2} = 0, & \frac{\partial \varphi^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{11}}{\partial z_3}-\frac{\partial \psi_{12}^{11}}{\partial z_2} = 0,\\
\frac{\partial \varphi^{11}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{21}}{\partial z_1}-\frac{\partial \psi_{23}^{21}}{\partial z_3} = 0, & \frac{\partial \varphi^{21}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{11}}{\partial z_1}-\frac{\partial \psi_{23}^{11}}{\partial z_3} = 0,\\
\frac{\partial \varphi^{11}}{\partial \bar{z}_3}+\frac{\partial \psi_{23}^{21}}{\partial z_2}+\frac{\partial \psi_{13}^{21}}{\partial z_1} = 0, & \frac{\partial \varphi^{21}}{\partial \bar{z}_3}+\frac{\partial \psi_{23}^{11}}{\partial z_2}+\frac{\partial \psi_{13}^{11}}{\partial z_1} = 0,\\
\frac{\partial \psi_3^{11}}{\partial \bar{z}_2}-\frac{\partial \psi_2^{11}}{\partial \bar{z}_3}+\frac{\partial \psi_{123}^{21}}{\partial z_1} = 0, & \frac{\partial \psi_3^{21}}{\partial \bar{z}_2}-\frac{\partial \psi_2^{21}}{\partial \bar{z}_3}+\frac{\partial \psi_{123}^{11}}{\partial z_1} = 0,\\
\frac{\partial \psi_3^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{11}}{\partial \bar{z}_3}-\frac{\partial \psi_{123}^{21}}{\partial z_2} = 0, & \frac{\partial \psi_3^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{21}}{\partial \bar{z}_3}-\frac{\partial \psi_{123}^{11}}{\partial z_2} = 0,\\
\frac{\partial \psi_2^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{11}}{\partial \bar{z}_2}+\frac{\partial \psi_{123}^{21}}{\partial z_3} = 0, & \frac{\partial \psi_2^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{21}}{\partial \bar{z}_2}+\frac{\partial \psi_{123}^{11}}{\partial z_3} = 0,\\
\frac{\partial \psi_{23}^{11}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{11}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{11}}{\partial \bar{z}_3} = 0, & \frac{\partial \psi_{23}^{21}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{21}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{21}}{\partial \bar{z}_3} = 0,
\end{cases}
\tag{3.4.1}
$$

$$
\begin{cases}
\frac{\partial \psi_1^{22}}{\partial z_1}+\frac{\partial \psi_2^{22}}{\partial z_2}+\frac{\partial \psi_3^{22}}{\partial z_3} = 0, & \frac{\partial \psi_1^{12}}{\partial z_1}+\frac{\partial \psi_2^{12}}{\partial z_2}+\frac{\partial \psi_3^{12}}{\partial z_3} = 0,\\
\frac{\partial \varphi^{12}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{22}}{\partial z_3}-\frac{\partial \psi_{12}^{22}}{\partial z_2} = 0, & \frac{\partial \varphi^{22}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{12}}{\partial z_3}-\frac{\partial \psi_{12}^{12}}{\partial z_2} = 0,\\
\frac{\partial \varphi^{12}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{22}}{\partial z_1}-\frac{\partial \psi_{23}^{22}}{\partial z_3} = 0, & \frac{\partial \varphi^{22}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{12}}{\partial z_1}-\frac{\partial \psi_{23}^{12}}{\partial z_3} = 0,\\
\frac{\partial \varphi^{12}}{\partial \bar{z}_3}+\frac{\partial \psi_{23}^{22}}{\partial z_2}+\frac{\partial \psi_{13}^{22}}{\partial z_1} = 0, & \frac{\partial \varphi^{22}}{\partial \bar{z}_3}+\frac{\partial \psi_{23}^{12}}{\partial z_2}+\frac{\partial \psi_{13}^{12}}{\partial z_1} = 0,\\
\frac{\partial \psi_3^{12}}{\partial \bar{z}_2}-\frac{\partial \psi_2^{12}}{\partial \bar{z}_3}+\frac{\partial \psi_{123}^{22}}{\partial z_1} = 0, & \frac{\partial \psi_3^{22}}{\partial \bar{z}_2}-\frac{\partial \psi_2^{22}}{\partial \bar{z}_3}+\frac{\partial \psi_{123}^{12}}{\partial z_1} = 0,\\
\frac{\partial \psi_3^{12}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{12}}{\partial \bar{z}_3}-\frac{\partial \psi_{123}^{22}}{\partial z_2} = 0, & \frac{\partial \psi_3^{22}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{22}}{\partial \bar{z}_3}-\frac{\partial \psi_{123}^{12}}{\partial z_2} = 0,\\
\frac{\partial \psi_2^{12}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{12}}{\partial \bar{z}_2}+\frac{\partial \psi_{123}^{22}}{\partial z_3} = 0, & \frac{\partial \psi_2^{22}}{\partial \bar{z}_1}-\frac{\partial \psi_1^{22}}{\partial \bar{z}_2}+\frac{\partial \psi_{123}^{12}}{\partial z_3} = 0,\\
\frac{\partial \psi_{23}^{12}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{12}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{12}}{\partial \bar{z}_3} = 0, & \frac{\partial \psi_{23}^{22}}{\partial \bar{z}_1}-\frac{\partial \psi_{13}^{22}}{\partial \bar{z}_2}+\frac{\partial \psi_{12}^{22}}{\partial \bar{z}_3} = 0.
\end{cases}
\tag{3.4.2}
$$

In particular they mean that:

φ^{11}, φ^{21}, φ^{12} and φ^{22} can be taken holomorphic in three variables with ψ^{11}_{123}, ψ^{21}_{123}, ψ^{12}_{123} and ψ^{22}_{123} taken antiholomorphic, while

ψ^{11}_3, ψ^{21}_3, ψ^{12}_3 and ψ^{22}_3 are taken holomorphic in the variables z_1, z_2 and antiholomorphic in the variable z_3 and ψ^{11}_2, ψ^{21}_2, ψ^{12}_2 and ψ^{22}_2 are taken holomorphic in the variables z_1, z_3 and antiholomorphic in the variable z_2;

and analogously,

ψ^{11}_1, ψ^{21}_1, ψ^{12}_1 and ψ^{22}_1 are taken holomorphic in the variables z_2, z_3 and antiholomorphic in the variable z_1 while ψ^{11}_{23}, ψ^{21}_{23}, ψ^{12}_{23} and ψ^{22}_{23} are taken holomorphic in the variable z_1 and antiholomorphic in the variables z_2, z_3;

and finally,

ψ^{11}_{13}, ψ^{21}_{13}, ψ^{12}_{13} and ψ^{22}_{13} are taken holomorphic in the variable z_2 and antiholomorphic in the variables z_1,z_3 and ψ^{11}_{12}, ψ^{21}_{12}, ψ^{12}_{12} and ψ^{22}_{12} are taken holomorphic in the variable z_3 and antiholomorphic in the variables z_1, z_2.

Again $\overline{\mathfrak{N}}(\Omega)$ contains, as proper subspaces:

1. the set $Hol\left(\Omega;\ \mathbb{C}^3\right)$ of all holomorphic mappings,

2. the set isomorphic to the set $\overline{Hol}\left(\Omega;\ \mathbb{C}^3\right)$ of all antiholomorphic mappings,

3. the sets isomorphic with the sets of mappings, whose coordinate functions are holomorphic with respect to some variables and antiholomorphic with respect to the others;

but $\overline{\mathfrak{N}}(\Omega)$ is not exhausted with them.

3.5 Hyperholomorphy for any number of variables

Consider now the case of an arbitrary m. We have

$$F^{\alpha\beta} = \varphi^{\alpha\beta} + \sum_{k=1}^{m} \sum_{|\mathbf{j}|=k} \psi_{\mathbf{j}}^{\alpha\beta} d\bar{z}^{\mathbf{j}}.$$

For $F^{\alpha\beta}$ one has

$$\begin{aligned} \bar{\partial}\left[F^{\alpha\beta}\right] &= \sum_{q=1}^{m} \frac{\partial \varphi^{\alpha\beta}}{\partial \bar{z}_q} d\bar{z}^q + \\ &\quad + \sum_{k=2}^{m} \sum_{|\mathbf{j}|=k} \sum_{q=1}^{k} (-1)^{q-1} \frac{\partial \psi^{\alpha\beta}_{j_1 \dots j_{q-1} j_{q+1} \dots j_k}}{\partial \bar{z}_{j_q}} d\bar{z}^{\mathbf{j}}, \end{aligned} \tag{3.5.1}$$

and

$$\begin{aligned} \bar{\partial}^*\left[F^{\alpha\beta}\right] &= \sum_{q=1}^{m} \frac{\partial \psi_q^{\alpha\beta}}{\partial z_q} + \\ &\quad + \sum_{k=1}^{m-1} \sum_{|\mathbf{j}|=k} \sum_{p=1}^{k+1} \sum_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \frac{\partial \psi^{\alpha\beta}_{j_1 \dots j_{p-1},\, q,\, j_p \dots j_k}}{\partial z_q} d\bar{z}^{\mathbf{j}}, \end{aligned} \tag{3.5.2}$$

where, by definition,

$$j_0 := 0, \qquad j_{k+1} := m + 1,$$

for any multiindex $\mathbf{j}$.

Thus $F = \left(F^{\alpha\beta}\right)_{1 \le \alpha \le 2}^{1 \le \beta \le 2}$ belongs to $\overline{\mathfrak{N}}(\Omega)$ in $\mathbb{C}^m$, with $2 \le m$, if and only if the following hyperholomorphic Cauchy-Riemann conditions hold:

$$
\begin{cases}
\sum\limits_{q=1}^{m} \frac{\partial \psi_q^{21}}{\partial z_q} & = 0, & \\
\\
\frac{\partial \varphi^{11}}{\partial \bar{z}_q} + \sum\limits_{p=1}^{q-1} \frac{\partial \psi_{pq}^{21}}{\partial z_p} - \sum\limits_{p=q+1}^{m} \frac{\partial \psi_{qp}^{21}}{\partial z_p} & = 0, & 2 \leq m, \\
& & q = 1, \ \ldots, \ m, \\
\sum\limits_{q=1}^{k} (-1)^{q-1} \frac{\partial \psi_{j_1 \cdots j_{q-1} j_{q+1} \cdots j_k}^{11}}{\partial \bar{z}_{j_q}} + & & \text{for } 3 \leq m, \\
& & k = 2, \ \ldots, \ m-1, \\
+ \sum\limits_{p=1}^{k+1} \sum\limits_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \frac{\partial \psi_{j_1 \cdots j_{p-1} q j_p \cdots j_k}^{21}}{\partial z_q} & = 0, & 1 \leq j_1 < \cdots < j_k \leq m, \\
\\
\sum\limits_{q=1}^{m} (-1)^{q-1} \frac{\partial \psi_{1 \ldots (q-1)(q+1) \ldots m}^{11}}{\partial \bar{z}_q} & = 0, & \\
\\
\sum\limits_{q=1}^{m} \frac{\partial \psi_q^{11}}{\partial z_q} & = 0, & \\
\\
\frac{\partial \varphi^{21}}{\partial \bar{z}_q} + \sum\limits_{p=1}^{q-1} \frac{\partial \psi_{pq}^{11}}{\partial z_p} - \sum\limits_{p=q+1}^{m} \frac{\partial \psi_{qp}^{11}}{\partial z_p} & = 0, & 2 \leq m, \\
& & q = 1, \ \ldots, \ m, \\
\sum\limits_{q=1}^{k} (-1)^{q-1} \frac{\partial \psi_{j_1 \cdots j_{q-1} j_{q+1} \cdots j_k}^{21}}{\partial \bar{z}_{j_q}} + & & \text{for } 3 \leq m, \\
& & k = 2, \ \ldots, \ m-1, \\
+ \sum\limits_{p=1}^{k+1} \sum\limits_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \frac{\partial \psi_{j_1 \cdots j_{p-1} q j_p \cdots j_k}^{11}}{\partial z_q} & = 0, & 1 \leq j_1 < \cdots < j_k \leq m, \\
\\
\sum\limits_{q=1}^{m} (-1)^{q-1} \frac{\partial \psi_{1 \ldots (q-1)(q+1) \ldots m}^{21}}{\partial \bar{z}_q} & = 0, &
\end{cases}
\tag{3.5.3}
$$

$$
\begin{cases}
\sum\limits_{q=1}^{m} \frac{\partial \psi_q^{22}}{\partial z_q} & = 0, & \\
\\
\frac{\partial \varphi^{12}}{\partial \bar{z}_q} + \sum\limits_{p=1}^{q-1} \frac{\partial \psi_{pq}^{22}}{\partial z_p} - \sum\limits_{p=q+1}^{m} \frac{\partial \psi_{qp}^{22}}{\partial z_p} & = 0, & 2 \leq m, \\
\& & q = 1, \ \ldots, \ m, & \\
\sum\limits_{q=1}^{k} (-1)^{q-1} \frac{\partial \psi_{j_1 \cdots j_{q-1} j_{q+1} \cdots j_k}^{12}}{\partial \bar{z}_{j_q}} + & & \text{for } 3 \leq m, \\
& & k = 2, \ \ldots, \ m-1, \\
+ \sum\limits_{p=1}^{k+1} \sum\limits_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \frac{\partial \psi_{j_1 \cdots j_{p-1} q j_p \cdots j_k}^{22}}{\partial z_q} & = 0, & 1 \leq j_1 < \cdots < j_k \leq m, \\
\\
\sum\limits_{q=1}^{m} (-1)^{q-1} \frac{\partial \psi_{1 \ldots (q-1)(q+1) \ldots m}^{12}}{\partial \bar{z}_q} & = 0, & \\
\\
\sum\limits_{q=1}^{m} \frac{\partial \psi_q^{12}}{\partial z_q} & = 0, & \\
\\
\frac{\partial \varphi^{22}}{\partial \bar{z}_q} + \sum\limits_{p=1}^{q-1} \frac{\partial \psi_{pq}^{12}}{\partial z_p} - \sum\limits_{p=q+1}^{m} \frac{\partial \psi_{qp}^{12}}{\partial z_p} & = 0, & 2 \leq m, \\
& & q = 1, \ \ldots, \ m, \\
\sum\limits_{q=1}^{k} (-1)^{q-1} \frac{\partial \psi_{j_1 \cdots j_{q-1} j_{q+1} \cdots j_k}^{22}}{\partial \bar{z}_{j_q}} + & & \text{for } 3 \leq m, \\
& & k = 2, \ \ldots, \ m-1, \\
+ \sum\limits_{p=1}^{k+1} \sum\limits_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \frac{\partial \psi_{j_1 \cdots j_{p-1} q j_p \cdots j_k}^{12}}{\partial z_q} & = 0, & 1 \leq j_1 < \cdots < j_k \leq m, \\
\\
\sum\limits_{q=1}^{m} (-1)^{q-1} \frac{\partial \psi_{1 \ldots (q-1)(q+1) \ldots m}^{22}}{\partial \bar{z}_q} & = 0, &
\end{cases}
\tag{3.5.4}
$$

where, by definition,

$$j_0 := 0, \qquad j_{k+1} := m + 1.$$

In particular, if $f : \Omega \to \mathbb{C}$ is a function of m complex variables, which is antiholomorphic in the "k" variables $z_{j_1}, \ldots, z_{j_k}$ ($k = 0, \ldots, m$) and holomorphic in the remaining variables, then the matrix

$$\begin{pmatrix} f d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_k} & 0 \\ 0 & f d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_k} \end{pmatrix} \in \overline{\mathfrak{N}}(\Omega).$$

This **m.v.d.f.** can serve a canonical representation of f as a 2×2-**m.v.d.f.**. Thus $\overline{\mathfrak{N}}(\Omega)$ contains, as proper subspaces, the set $Hol(\Omega; \mathbb{C})$ of all holomorphic functions, the set isomorphic to the set $\overline{Hol}(\Omega; \mathbb{C})$ of all antiholomorphic functions, as well as other sets analogous to those described for $m = 2$ and $m = 3$. Obviously, $\overline{\mathfrak{N}}(\Omega)$ can be described as a set of mappings $\Omega \to \mathbb{C}^{4 \cdot 2^m}$ (generated by coordinates of elements from $\overline{\mathfrak{N}}(\Omega)$), and it is important to note that among the coordinates of each element, there are not more than four holomorphic functions, not more than four antiholomorphic functions, and not more than four functions from any other combination of holomorphy with respect to some variables and antiholomorphy with respect to the rest of them. But $\overline{\mathfrak{N}}(\Omega)$ is not exhausted with them.

3.6 Observation about right-hand-side hyperholomorphy

We are not going to describe in detail the subclasses of $\overline{\mathfrak{M}}_{r;\,G}(\Omega)$, just several observations.

Let here $m = 1$, i.e., we are interested in the class $\overline{\mathfrak{M}}_{r;\,G}(\Omega)$ in $\mathbb{C} = \mathbb{C}^1$, where Ω is a connected domain in $\mathbb{C}$. Recall that $\overline{\mathcal{D}}_r[F]$ is not a **m.v.d.f.,** it is an operator acting on **m.v.d.f.** Let now $G \in \overline{\mathfrak{G}}_0(\Omega)$, then

$$\overline{\mathcal{D}}_r[F] \underset{\star}{\wedge} G = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}$$

with

$$\begin{aligned}
\nu_{11} &= \frac{\partial\varphi^{11}}{\partial\bar{z}}g_0^{11}d\bar{z}+\frac{\partial\varphi^{12}}{\partial z}g_1^{11}+\frac{\partial\psi^{12}}{\partial z}g_1^{11}d\bar{z}+\\
&\quad+\frac{\partial\varphi^{12}}{\partial\bar{z}}g_0^{21}d\bar{z}+\frac{\partial\varphi^{11}}{\partial z}g_1^{21}+\frac{\partial\psi^{11}}{\partial z}g_1^{21}d\bar{z};\\
\nu_{12} &= \frac{\partial\varphi^{11}}{\partial\bar{z}}g_0^{12}d\bar{z}+\frac{\partial\varphi^{12}}{\partial z}g_1^{12}+\frac{\partial\psi^{12}}{\partial z}g_1^{12}d\bar{z}+\\
&\quad+\frac{\partial\varphi^{12}}{\partial\bar{z}}g_0^{22}d\bar{z}+\frac{\partial\varphi^{11}}{\partial z}g_1^{22}+\frac{\partial\psi^{12}}{\partial z}g_1^{22}d\bar{z};\\
\nu_{21} &= \frac{\partial\varphi^{21}}{\partial\bar{z}}g_0^{11}d\bar{z}+\frac{\partial\varphi^{22}}{\partial z}g_1^{11}+\frac{\partial\psi^{22}}{\partial z}g_1^{11}d\bar{z}+\\
&\quad+\frac{\partial\varphi^{22}}{\partial\bar{z}}g_0^{21}d\bar{z}+\frac{\partial\varphi^{21}}{\partial z}g_1^{21}+\frac{\partial\psi^{21}}{\partial z}g_1^{21}d\bar{z};\\
\nu_{22} &= \frac{\partial\varphi^{21}}{\partial\bar{z}}g_0^{12}d\bar{z}+\frac{\partial\varphi^{22}}{\partial z}g_1^{12}+\frac{\partial\psi^{22}}{\partial z}g_1^{12}d\bar{z}+\\
&\quad+\frac{\partial\varphi^{22}}{\partial\bar{z}}g_0^{22}d\bar{z}+\frac{\partial\varphi^{21}}{\partial z}g_1^{22}+\frac{\partial\psi^{11}}{\partial z}g_1^{22}d\bar{z},
\end{aligned}$$

where $F\in\overline{\mathfrak{G}}_1(\Omega)$; so that any $F^{\alpha\beta}$ is of the form $\varphi^{\alpha\beta}+\psi^{\alpha\beta}d\bar{z}$ for $\alpha,\ \beta\in\{1,\ 2\}$, and $\varphi^{\alpha\beta}$ and $\psi^{\alpha\beta}$ are from $C^1(\Omega;\ \mathbb{C})$; $G^{\alpha\beta}:=g_0^{\alpha\beta}+g_1^{\alpha\beta}d\bar{z}$.

Note first that $\overline{\mathcal{D}}_r[F]\equiv 0$ iff $\varphi^{\alpha\beta}$ are constant functions and $\psi^{\alpha\beta}$ are antiholomorphic functions.

Suppose that all coordinates of G are equal to zero less g_0^{11}, then $F\in\overline{\mathfrak{M}}_{r;\,G}(\Omega)$ iff φ^{11} and φ^{21} are holomorphic functions and the rest of the coordinates are only differentiable functions, so there is no Cauchy's integral theorem for that F.

Now, suppose that all coordinates of G are equal to zero less g_1^{11} and g_1^{21}, with $g_1^{11}\equiv g_1^{21}$; then $F\in\overline{\mathfrak{M}}_{r;\,G}(\Omega)$ iff $\frac{\partial\varphi^{12}}{\partial z}+\frac{\partial\varphi^{11}}{\partial z}\equiv 0$ and $\frac{\partial\psi^{12}}{\partial z}+\frac{\partial\psi^{11}}{\partial z}\equiv 0$, and this is not the one-variable complex analysis.

The set $\overline{\mathfrak{M}}_{r;\,G}(\Omega)$ is only auxiliary for us which we need in order to have a symmetric Morera's theorem, and it is quite different from traditional theory.

Chapter 4

Hyperholomorphic Cauchy's integral theorems

4.1 The Cauchy integral theorem for left-hyperholomorphic matrix-valued differential forms

Theorem *Let Ω be a domain with the topological boundary Γ, which is a piecewise smooth surface, let $G \in \overline{\mathfrak{N}}(\Omega) \cap \overline{\mathfrak{G}}_0(\Omega \cup \Gamma)$. Then the following equality holds:*

$$\int_{\Gamma} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} G(\tau,\ d\bar{z}) = 0. \tag{4.1.1}$$

Proof. It is enough to look at the Stokes formula, see Theorem 2.7. ■

4.2 The Cauchy integral theorem for right-G-hyperholomorphic m.v.d.f.

Theorem *Let Ω be a domain with the topological boundary Γ, which is a piecewise smooth surface, let $G \in \overline{\mathfrak{N}}(\Omega) \cap \overline{\mathfrak{G}}_0(\Omega \cup \Gamma)$ and $F \in \overline{\mathfrak{M}}_{r;\, G}(\Omega) \cap \overline{\mathfrak{G}}_0(\Omega \cup \Gamma)$. Then the following equalities hold:*

$$\int_{\Gamma} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} G(\tau,\ d\bar{z}) = 0,$$

$$\int\limits_{\Gamma} F\left(\tau,\ d\bar{z}\right) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} G\left(\tau,\ d\bar{z}\right) = 0.$$

Proof. The second equality is again a corollary of the Stokes formula from Theorem 2.7. ■

We consider now several corollaries for conventional complex analysis in $\mathbb{C}^m$.

4.3 Some auxiliary computations

Let first $G = gE_{2\times 2}$ where g is a holomorphic function of several complex variables. Then G satisfies the formula (4.1.1), hence

$$\int\limits_{\Gamma} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} gE_{2\times 2} = 0,$$

that is,

$$\int\limits_{\Gamma} \begin{pmatrix} \bar{\sigma}\wedge g & \bar{\sigma}^*\wedge g \\ \bar{\sigma}^*\wedge g & \bar{\sigma}\wedge g \end{pmatrix} = 0,$$

which is equivalent to

$$\int\limits_{\Gamma} \bar{\sigma}\wedge g = 0.$$

More explicitly, the last equality has the form

$$\sum_{j=1}^{m} \left(\int\limits_{\Gamma} g\left(\zeta\right) d\bar{\zeta}_{[j]} \wedge d\zeta \right) d\bar{z}^j = 0;$$

which is equivalent to any of two systems of equalities:

$$\int\limits_{\Gamma} g\left(\zeta\right) d\bar{\zeta}_{[j]} \wedge d\zeta \ = \ 0, \qquad \text{for any } j = 1,\ \ldots,\ m,$$

and

$$\int\limits_{\Gamma} n_j\left(\zeta\right) g\left(\zeta\right) dS_{\zeta} \ = \ 0, \qquad \text{for any } j = 1,\ \ldots,\ m,$$

where $\mathbf{n}(\zeta) = (n_1(\zeta), \ldots, n_m(\zeta))$ is the unit outward-pointing normal vector to Γ at ζ.

4.4 More auxiliary computations

The above reasonings allow an immediate generalization.

Let $G = g d\bar{z}^{\mathbf{j}} E_{2\times 2}$ where g is a complex function, which is antiholomorphic in the variables $z_{j_1}, \ldots, z_{j_{|\mathbf{j}|}}$ and holomorphic in the rest of variables, with $\mathbf{j} := \left(j_1, \ldots, j_{|\mathbf{j}|}\right)$ a strictly increasing $|\mathbf{j}|$-tuple in $\{1, \ldots, m\}$. Then G satisfies the formula (4.1.1), hence

$$\int_{\Gamma} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} g d\bar{z}^{\mathbf{j}} E_{2\times 2} = 0,$$

that is,

$$\int_{\Gamma} \begin{pmatrix} \bar{\sigma} \wedge g d\bar{z}^{\mathbf{j}} & \bar{\sigma}^* \wedge g d\bar{z}^{\mathbf{j}} \\ \bar{\sigma}^* \wedge g d\bar{z}^{\mathbf{j}} & \bar{\sigma} \wedge g d\bar{z}^{\mathbf{j}} \end{pmatrix} = 0,$$

which is equivalent to

$$\begin{aligned} \sum_{j=1}^{m} \left(\int_{\Gamma} g\left(\zeta\right) d\bar{\zeta}_{[j]} \wedge d\zeta \right) d\bar{z}^{j} \wedge d\bar{z}^{\mathbf{j}} &= 0, \\ \sum_{j=1}^{m} \left(\int_{\Gamma} g\left(\zeta\right) d\bar{\zeta} \wedge d\zeta_{[j]} \right) \widehat{d\bar{z}^{j}} \wedge d\bar{z}^{\mathbf{j}} &= 0. \end{aligned}$$

More explicitly, the last equalities have the form

$$\int_{\Gamma} g\left(\zeta\right) d\bar{\zeta}_{[j]} \wedge d\zeta = 0, \text{ for } j = 1, \ldots, m \text{ and } j \neq j_1, \ldots, j_{|\mathbf{j}|},$$

$$\int_{\Gamma} g\left(\zeta\right) d\bar{\zeta} \wedge d\zeta_{[j]} = 0, \text{ for any } j = j_1, \ldots, j_{|\mathbf{j}|},$$

or equivalently,

$$\int_{\Gamma} n_j\left(\zeta\right) g\left(\zeta\right) dS_{\zeta} = 0, \text{ for } j = 1, \ldots, m \text{ and } j \neq j_1, \ldots, j_{|\mathbf{j}|},$$

$$\int_{\Gamma} \bar{n}_j\left(\zeta\right) g\left(\zeta\right) dS_{\zeta} = 0, \text{ for any } j = j_1, \ldots, j_{|\mathbf{j}|},$$

where $\overline{n}_j$ is the complex conjugate of the j-th coordinate of the vector $\mathbf{n}$, dS_{ζ} is the element of surface.

We resume the above computations in the following corollaries.

4.5 The Cauchy integral theorem for holomorphic functions of several complex variables

Corollary *Let g be a function holomorphic in the bounded domain Ω^+ and continuous in $\Omega^+ \cup \Gamma$ with Γ being a piecewise smooth boundary of Ω^+. Then each of two equivalent conditions holds:*

1. $$\int\limits_{\Gamma} g(\zeta)\, d\bar{\zeta}_{[j]} \wedge d\zeta = 0, \qquad \text{for any } j = 1, \ldots, m,$$

2. $$\int\limits_{\Gamma} n_j(\zeta)\, g(\zeta)\, dS_\zeta = 0, \qquad \text{for any } j = 1, \ldots, m,$$

which are also expressed in slightly different terms:

$1'$. $g|\Gamma$ is orthogonal to each **d.f.** $d\bar{\zeta}_{[j]} \wedge d\zeta$, *where* $j = 1, \ldots, m$,

$2'$. g is orthogonal to the normal vector $\mathbf{n}$:

$$\int\limits_{\Gamma} g(\zeta)\, \mathbf{n}(\zeta)\, dS_\zeta = \mathbf{0},$$

where it is clear what the "orthogonality" means.

4.6 The Cauchy integral theorem for antiholomorphic functions of several complex variables

Corollary *Let g be a function antiholomorphic in the bounded domain Ω^+ and continuous in $\Omega^+ \cup \Gamma$ with Γ being a piecewise smooth boundary of Ω^+. Then each of two equivalent conditions holds:*

1. $$\int\limits_{\Gamma} g(\zeta)\, d\bar{\zeta} \wedge d\zeta_{[j]} = 0, \qquad \text{for any } j = 1, \ldots, m,$$

2. $$\int\limits_{\Gamma} \bar{n}_j(\zeta)\, g(\zeta)\, dS_\zeta = 0, \qquad \text{for any } j = 1, \ldots, m,$$

which are expressed also in slightly different terms:

1'. $g|\Gamma$ is orthogonal to each **d.f.** $d\bar{\zeta} \wedge d\zeta_{[j]}$, *where* $j = 1, \ldots, m$.

2'. g is orthogonal to the complex conjugate of the vector $\mathbf{n}$:

$$\int_{\Gamma} g(\zeta)\, \bar{\mathbf{n}}(\zeta)\, dS_{\zeta} = \mathbf{0}.$$

4.7 The Cauchy integral theorem for functions holomorphic in some variables and antiholomorphic in the rest of variables

Corollary *Let g be a function antiholomorphic in the bounded domain Ω^+ in the variables $z_{j_1}, \ldots, z_{j_{|\mathbf{j}|}}$, holomorphic in the rest of variables, and continuous in $\Omega^+ \cup \Gamma$. Then each of two equivalent conditions holds:*

1.

$$\int_{\Gamma} g(\zeta)\, d\bar{\zeta}_{[j]} \wedge d\zeta = 0, \qquad \text{for } j = 1, \ldots, m \text{ and } j \neq j_1, \ldots, j_{|\mathbf{j}|},$$

$$\int_{\Gamma} g(\zeta)\, d\bar{\zeta} \wedge d\zeta_{[j]} = 0, \qquad \text{for any } j = j_1, \ldots, j_{|\mathbf{j}|},$$

2.

$$\int_{\Gamma} n_j(\zeta)\, g(\zeta)\, dS_{\zeta} = 0, \qquad \text{for } j = 1, \ldots, m \text{ and } j \neq j_1, \ldots, j_{|\mathbf{j}|},$$

$$\int_{\Gamma} \bar{n}_j(\zeta)\, g(\zeta)\, dS_{\zeta} = 0, \qquad \text{for any } j = j_1, \ldots, j_{|\mathbf{j}|},$$

which are also expressed in slightly different terms:

1'. $g|\Gamma$ is orthogonal to each **d.f.** $d\bar{\zeta} \wedge d\zeta_{[j]}$, *for j in $\{j_1, \ldots, j_{|\mathbf{j}|}\}$ and to each* **d.f.** $d\bar{\zeta}_{[j]} \wedge d\zeta$ *for $j \notin \{j_1, \ldots, j_m\}$.*

2'. g is orthogonal to the vector $\tilde{\mathbf{n}} := (\tilde{n}_1, \ldots, \tilde{n}_m)$ with $\tilde{n}_j = \overline{n}_j$ for $j \in \{j_1, \ldots, j_{|\mathbf{j}|}\}$ and $\tilde{n}_j = n_j$ for the remaining indices.

4.8 Concluding remarks

Of course Corollaries 4.5, 4.6 and 4.7 do not pretend to be great novelties, one can find Corollary 4.5 in many sources, and Corollaries 4.6 and 4.7 can be extracted from it. But we are trying again to stress their relation to the hyperholomorphic theory. What is more, although Corollaries 4.5 and 4.6 are of course particular cases of Corollary 4.7, there is a slight difference between Corollary 4.5 on one hand, and Corollaries 4.6 and 4.7 that reflects the structure of the set $\overline{\mathfrak{N}}(\Omega)$ (see remarks concluding Subsection 3.5): Corollary 4.5 presents nothing more than the equality (4.1.1) with a specific matrix $G(\tau, d\bar{z})$, that is, Corollary 4.5 is exactly the equality (4.1.1) restricted to a subclass of hyperholomorphic matrices; while Corollaries 4.6 and 4.7 are not, exactly speaking, specific cases of the equality (4.1.1), they give a property of coefficients of certain **m.v.d.f.** which satisfy (4.1.1).

Anyhow it is an insignificant abuse to say that all three corollaries are contained directly in the equality (4.1.1), or that they are just restrictions of (4.1.1) onto different subclasses of $\overline{\mathfrak{N}}(\Omega)$, and one may consider more subclasses.

Chapter 5

Hyperholomorphic Morera's theorems

5.1 Left-hyperholomorphic Morera theorem

Theorem *Let Ω^{+} be a domain in $\mathbb{C}^{m}$ with the topological boundary Γ, let $G \in \overline{\mathfrak{G}}_{1}\left(\Omega^{+}\right)$. If for any bounded surface-without-boundary $\hat{\Gamma}$ which is a piecewise smooth surface, with $\hat{\Gamma} = Fr\left(\hat{\Omega}^{+}\right)$, $\hat{\Omega}^{+} \subset \Omega^{+}$, $\hat{\Gamma} \subset \Omega^{+}$, the following equality holds:*

$$\int\limits_{\hat{\Gamma}} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} G\left(\tau,\ d\bar{z}\right) = 0, \tag{5.1.1}$$

then

$$G \in \overline{\mathfrak{N}}\left(\Omega^{+}\right).$$

Proof. Let $z \in \Omega^{+}$ be an arbitrary point, $\left\{\mathbb{B}\left(z;\ \varepsilon_{k}\right)\right\}_{k=1}^{\infty}$ be a regular sequence of balls which is contracting to z, we suppose that, for any $k \in \mathbb{N}$, $\mathbb{B}\left(z;\ \varepsilon_{k}\right) \cup \mathbb{S}\left(z;\ \varepsilon_{k}\right) \subset \Omega$. By Lebesgue's theorem we obtain for the **m.v.d.f.** $\overline{\mathcal{D}}\left[G\right]$, which is continuous and bounded on $\mathbb{B}\left(z;\ \varepsilon_{k}\right) \cup \mathbb{S}\left(z;\ \varepsilon_{k}\right)$, that

$$\lim_{k\to\infty} \frac{1}{\left|\mathbb{B}\left(z;\ \varepsilon_{k}\right)\right|} \int\limits_{\mathbb{B}\left(z;\ \varepsilon_{k}\right)} \overline{\mathcal{D}}\left[G\right]\left(\tau,\ d\bar{z}\right) dV_{\tau} = \overline{\mathcal{D}}\left[G\right]\left(z\right), \tag{5.1.2}$$

where $|\mathbb{B}(z;\ \varepsilon_k)|$ denotes the usual volume of $\mathbb{B}(z;\ \varepsilon_k)$. Note that the formula (2.8.1) takes the form

$$\int\limits_{\mathbb{B}(z;\ \varepsilon_k)} \overline{\mathcal{D}}[G](\tau,\ d\bar{z})\, dV_\tau = \int\limits_{\mathbb{S}(z;\ \varepsilon_k)} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} G(\tau,\ d\bar{z}). \tag{5.1.3}$$

Substituting (5.1.3) into (5.1.2), we have

$$\lim_{k\to\infty} \frac{1}{|\mathbb{B}(z;\ \varepsilon_k)|} \int\limits_{\mathbb{S}(z;\ \varepsilon_k)} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} G(\tau,\ d\bar{z}) = \overline{\mathcal{D}}[G](z). \tag{5.1.4}$$

Substituting (5.1.1) into (5.1.4), we have

$$\overline{\mathcal{D}}[G](z) = 0.$$

But z is an arbitrary point of Ω.

Hence $G \in \overline{\mathfrak{N}}(\Omega)$. ■

Of course, the above theorem can be considered as an inverse one to some reformulation of Theorem 4.1; thus we obtain an equivalent definition of hyperholomorphy in terms of vanishing integrals of a given **m.v.d.f.**

5.2 Version of a right-hyperholomorphic Morera theorem

Theorem *Let Ω be a domain in $\mathbb{C}^m$ with the boundary Γ, let $F,\ G \in \overline{\mathfrak{G}}_1(\Omega)$. If for any bounded surface-without-boundary $\hat{\Gamma}$ which is a piecewise smooth surface, with $\hat{\Gamma} = Fr\left(\hat{\Omega}\right)$, $\hat{\Omega} \subset \Omega$, $\hat{\Gamma} \subset \Omega$, the following equalities hold:*

$$\int\limits_{\hat{\Gamma}} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} G(\tau,\ d\bar{z}) = 0, \tag{5.2.1}$$

$$\int\limits_{\hat{\Gamma}} F(\tau,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z} \underset{\star}{\wedge} G(\tau,\ d\bar{z}) = 0, \tag{5.2.2}$$

then

$$G \in \overline{\mathfrak{N}}\,(\Omega)\,,$$

$$F \in \overline{\mathfrak{M}}_{r;\,G}\,(\Omega)\,.$$

Proof. First, by Theorem 5.1, $G \in \overline{\mathfrak{N}}\,(\Omega)$; this is

$$\overline{\mathcal{D}}\,[G] = 0. \tag{5.2.3}$$

We will use the same notations as in the proof of Theorem 5.1. By Lebesgue's theorem we obtain for the **m.v.d.f.** $\overline{\mathcal{D}}_r\,[F] \underset{\star}{\wedge} G$, which is continuous and bounded on $\mathbb{B}\,(z;\ \varepsilon_k) \cup \mathbb{S}\,(z;\ \varepsilon_k)$, that

$$\begin{gathered}\lim_{k\to\infty} \tfrac{1}{|\mathbb{B}(z;\ \varepsilon_k)|} \int\limits_{\mathbb{B}(z;\ \varepsilon_k)} \overline{\mathcal{D}}_r\,[F]\,(\tau,\ d\bar{z}) \underset{\star}{\wedge} G\,(\tau,\ d\bar{z})\,dV_\tau = \\ = \overline{\mathcal{D}}_r\,[G]\,(z) \underset{\star}{\wedge} G\,(z)\,.\end{gathered} \tag{5.2.4}$$

Note that by (5.2.3) the formula (2.8.1) takes the form

$$\begin{gathered}\int\limits_{\mathbb{B}(z;\ \varepsilon_k)} \overline{\mathcal{D}}_r\,[F]\,(\tau,\ d\bar{z}) \underset{\star}{\wedge} G\,(\tau,\ d\bar{z})\,dV_\tau = \\ = \int\limits_{\mathbb{S}(z;\ \varepsilon_k)} F\,(\tau,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\,z} \underset{\star}{\wedge} G\,(\tau,\ d\bar{z})\,.\end{gathered} \tag{5.2.5}$$

Substituting (5.2.5) into (5.2.4), we have

$$\begin{gathered}\lim_{k\to\infty} \tfrac{1}{|\mathbb{B}(z;\ \varepsilon_k)|} \int\limits_{\mathbb{S}(z;\ \varepsilon_k)} F\,(\tau,\ d\bar{z}) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\,z} \underset{\star}{\wedge} G\,(\tau,\ d\bar{z}) = \\ = \overline{\mathcal{D}}_r\,[F]\,(z) \underset{\star}{\wedge} G\,(z)\,.\end{gathered} \tag{5.2.6}$$

Substituting (5.2.2) into (5.2.6), we have

$$\overline{\mathcal{D}}_r\,[F]\,(z) \underset{\star}{\wedge} G\,(z) = 0.$$

Again taking into account that z is an arbitrary point of Γ we obtain that $F \in \overline{\mathfrak{M}}_{r;\,G}\,(\Omega)$. ■

This is again an inverse theorem, now to Theorem 4.2, which means that we obtain an equivalent definition, in terms of vanishing integrals of two given **m.v.d.f.,** of the right-hand-side hyperholomorphy of one of them with respect to the **left-hyperholomorphic** (not arbitrary) second matrix.

5.3 Morera's theorem for holomorphic functions of several complex variables

Corollary *Let Ω be a domain in $\mathbb{C}^m$ with the topological boundary Γ, let g be a complex function of class C^1 in the domain Ω. If for any bounded surface-without-boundary $\hat{\Gamma}$ which is a piecewise smooth surface, with $\hat{\Gamma} = Fr\left(\hat{\Omega}\right)$, $\hat{\Omega} \subset \Omega$, $\hat{\Gamma} \subset \Omega$, one of the following equivalent conditions holds:*

1. $$\int\limits_{\hat{\Gamma}} g(\zeta)\, d\bar{\zeta}_{[j]} \wedge d\zeta = 0, \qquad \text{for any } j = 1, \ldots, m,$$

   (i.e., $g|\hat{\Gamma}$ is orthogonal to each **d.f.** *$d\bar{\zeta}_{[j]} \wedge d\zeta$, where $j = 1, \ldots, m$, with respect to the bilinear form defined by $(g, w) := \int\limits_{\hat{\Gamma}} g(\zeta)\, w(\zeta)$).*

2. $$\int\limits_{\hat{\Gamma}} n_j(\zeta)\, g(\zeta)\, dS_\zeta = 0, \qquad \text{for any } j = 1, \ldots, m,$$

   (i.e., $\int\limits_{\hat{\Gamma}} g(\zeta)\, \mathbf{n}(\zeta)\, dS_\zeta = \mathbf{0}$, the orthogonality of g and $\mathbf{n}$ on $\hat{\Gamma}$),

then g is a holomorphic function in the sense of several complex variables function theory.

Proof. Using the computation in Subsection 4.3 and combining it with Theorem 5.1, we have that $G := gE_{2\times 2}$ is in $\overline{\mathfrak{N}}(\Omega)$, this is:

$$\begin{pmatrix} \bar{\partial}[g] & \bar{\partial}^*[g] \\ \bar{\partial}^*[g] & \bar{\partial}[g] \end{pmatrix} = \overline{\mathcal{D}}[G] = 0,$$

or equivalently,

$$\bar{\partial}\left[g\right]=\sum_{j=1}^{m}\frac{\partial g}{\partial\bar{z}_{j}}d\bar{z}^{j}=0.$$

Then g is a holomorphic function in the sense of several complex variables function theory. ■

Of course, the Morera theorem for holomorphic functions has been long known; one can find an extensive literature on the subject.

5.4 Morera's theorem for antiholomorphic functions of several complex variables

Corollary *Let Ω be a domain in $\mathbb{C}^{m}$ with the topological boundary Γ, let g be a complex function of class C^{1} in the domain Ω. If for any bounded surface-without-boundary $\hat{\Gamma}$ which is a piecewise smooth surface, with $\hat{\Gamma}=Fr\left(\hat{\Omega}\right)$, $\hat{\Omega}\subset\Omega$, $\hat{\Gamma}\subset\Omega$, one of the following equivalent conditions holds:*

1. $$\int\limits_{\hat{\Gamma}}g\left(\zeta\right)d\bar{\zeta}\wedge d\zeta_{[j]}=0,\qquad\text{for any }j=1,\ \ldots,\ m,$$

   (i.e., $g|\hat{\Gamma}$ is orthogonal to each **d.f.** *$d\bar{\zeta}\wedge d\zeta_{[j]}$, where $j=1,\ \ldots,\ m$, with respect to the bilinear form defined by $(g,\ w):=\int\limits_{\hat{\Gamma}}g\left(\zeta\right)w\left(\zeta\right)$).*

2. $$\int\limits_{\hat{\Gamma}}\bar{n}_{j}\left(\zeta\right)g\left(\zeta\right)dS_{\zeta}=0,\qquad\text{for any }j=1,\ \ldots,\ m,$$

   (i.e., $\int\limits_{\hat{\Gamma}}g\left(\zeta\right)\bar{\mathbf{n}}\left(\zeta\right)dS_{\zeta}=\mathbf{0}$, the orthogonality of g and $\overline{\mathbf{n}}$ on $\hat{\Gamma}$),

then g is an antiholomorphic function.

Proof. Using the computation in Subsection 4.4 and combining it with Theorem 5.1, we have that $G:=gE_{2\times2}d\bar{z}$ is in $\overline{\mathfrak{N}}\left(\Omega\right)$, that is

$$\begin{pmatrix}\bar{\partial}\left[gd\bar{z}\right]; & \bar{\partial}^{*}\left[gd\bar{z}\right]\\ \bar{\partial}^{*}\left[gd\bar{z}\right]; & \bar{\partial}\left[gd\bar{z}\right]\end{pmatrix}=\overline{\mathcal{D}}\left[G\right]=0,$$

or equivalently,

$$\bar{\partial}^{*}\left[g d\bar{z}\right]=\sum_{j=1}^{m}\left(-1\right)^{j-1}\frac{\partial g}{\partial z_{j}}d\bar{z}_{[j]}=0.$$

Then g is an antiholomorphic function. ■

Formally, one can obtain a proof of Corollary 5.4 directly from Corollary 5.3. We have chosen, again, another way related to antiholomorphic functions as a subset of hyperholomorphic ones.

As in Chapter 4, both corollaries are included in a more general assertion.

5.5 The Morera theorem for functions holomorphic in some variables and antiholomorphic in the rest of variables

Corollary *Let Ω be a domain in $\mathbb{C}^{m}$ with the topological boundary Γ, let g be a complex-valued function of class C^{1} in the domain Ω and let $1\leq j_{1}<\cdots<j_{s}\leq m$, for some $s=0,\ \cdots,\ m$. If for any bounded surface-without-boundary $\hat{\Gamma}$ which is a piecewise smooth surface, with $\hat{\Gamma}=Fr\left(\hat{\Omega}\right)$, $\hat{\Omega}\subset\Omega$, $\hat{\Gamma}\subset\Omega$, one of the following equivalent conditions holds:*

1.

$$\int_{\Gamma}g\left(\zeta\right)d\bar{\zeta}_{[j]}\wedge d\zeta=0,\quad \text{for } j=1,\ \ldots,\ m \text{ and } j\neq j_{1},\ \ldots,\ j_{|\mathbf{j}|},$$

$$\int_{\Gamma}g\left(\zeta\right)d\bar{\zeta}\wedge d\zeta_{[j]}=0,\quad \text{for any } j=j_{1},\ \ldots,\ j_{|\mathbf{j}|},$$

2.

$$\int_{\Gamma}n_{j}\left(\zeta\right)g\left(\zeta\right)dS_{\zeta}=0,\quad \text{for } j=1,\ \ldots,\ m \text{ and } j\neq j_{1},\ \ldots,\ j_{|\mathbf{j}|},$$

$$\int_{\Gamma}\bar{n}_{j}\left(\zeta\right)g\left(\zeta\right)dS_{\zeta}=0,\quad \text{for any } j=j_{1},\ \ldots,\ j_{|\mathbf{j}|},$$

then g is antiholomorphic in the variables $z_{j_1}, \ldots, z_{j_s}$ and holomorphic in the rest of the variables.

Proof. Let $\mathbf{j} := (j_1, \ldots, j_s)$. Using the computation in Subsection 4.4 and combining it with Theorem 5.1, we have that $G := gE_{2\times 2}d\bar{z}^{\mathbf{j}}$ is in $\overline{\mathfrak{N}}(\Omega)$, that is

$$\overline{\mathcal{D}}[G] = \begin{pmatrix} \bar{\partial}\left[gd\bar{z}^{\mathbf{j}}\right] & \bar{\partial}^*\left[gd\bar{z}^{\mathbf{j}}\right] \\ \\ \bar{\partial}^*\left[gd\bar{z}^{\mathbf{j}}\right] & \bar{\partial}\left[gd\bar{z}^{\mathbf{j}}\right] \end{pmatrix} = 0,$$

equivalently,

$$\sum_{k=0}^{s}\sum_{j_k<j<j_{k+1}} (-1)^k \frac{\partial g}{\partial \bar{z}_j} d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_k} \wedge d\bar{z}^{j} \wedge d\bar{z}^{j_{k+1}} \wedge \ldots \wedge$$

$$\wedge d\bar{z}^{j_s} = 0,$$

$$\sum_{k=1}^{s} (-1)^{k-1} \frac{\partial g}{\partial z_{j_k}} d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_{k-1}} \wedge d\bar{z}^{j_{k+1}} \wedge \ldots \wedge d\bar{z}^{j_s} = 0.$$

Hence g is antiholomorphic in the variables $z_{j_1}, \ldots, z_{j_s}$ and holomorphic in the rest of the variables. ■

Again, a proof could have been obtained from Corollary 5.3, which we have not done in view of the above reasons.

Remark also that, abusing a little, as has been explained in Subsection 2.10 for the Cauchy integral theorem, Corollaries 5.3, 5.4 and 5.5 are contained directly in hyperholomorphic Morera's theorem 5.1, and that they are restrictions of Theorem 5.1 onto different subclasses of $\overline{\mathfrak{N}}(\Omega)$, and one may consider more subclasses.

Chapter 6

Hyperholomorphic Cauchy's integral representations

6.1 Cauchy's integral representation for left-hyperholomorphic matrix-valued differential forms

Theorem *Let Ω^{+} be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $F \in \overline{\mathfrak{N}}\left(\Omega^{+}\right) \cap \overline{\mathfrak{G}}_{0}\left(\Omega^{+} \cup \Gamma\right)$. Then the following equality holds in Ω^{+}:*

$$2F\left(z\right) = \int\limits_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}}\left(\tau - z,\ z\right) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F\left(\tau\right). \tag{6.1.1}$$

It follows trivially from the Borel-Pompeiu formula. As usual, the Cauchy integral representation for a class of functions has numerous consequences. But first we consider, as we did in Chapters 4 and 5, some corollaries of (6.1.1) for conventional complex analysis, i.e., its restrictions onto some subclasses of hyperholomorphic **m.v.d.f.**

6.2 A consequence for holomorphic functions

Let $G = gE_{2\times 2}$ where g is a holomorphic function of several complex variables. Then G satisfies the formula (6.1.1) and combining it with Subsection 2.10, we have

$$g(z) = (-1)^{\frac{m(m-1)}{2}} \int_{\Gamma} \mathcal{U}(\tau,\ z)\, g(\tau)\,, \tag{6.2.1}$$

$$0 = \overline{W}_{jk}[g](z) - \overline{W}_{kj}[g](z)\,, \tag{6.2.2}$$

where $j,\ k = 1,\ \ldots,\ m$, $j < k$, and the definitions of $\overline{W}_{jk}[g](z)$, $\overline{W}_{kj}[g](z)$ are in Subsection 2.11.8.

Note that the equality (6.2.1) is the usual Bochner-Martinelli integral representation of the holomorphic function g. There is a well-known paradox of the Bochner-Martinelli integral: its kernel $\mathcal{U}$ is non-holomorphic, but it reproduces, by (6.2.1), the holomorphic function g; what is more, given density g (not necessarily the boundary value of a holomorphic function), the Bochner-Martinelli-type integral is holomorphic if and only if the density g is the boundary value of a holomorphic function (see, for instance, [Ky]). Theorem 6.1 explains, in a certain sense, why it occurs: being non-holomorphic, the kernel $\mathcal{U}$ is generated by the product $\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\ z}$ in (6.1.1), which reproduces **all** hyperholomorphic **m.v.d.f.,** and (6.2.1) is nothing more (from this point of view) than the restriction of (6.1.1) onto a specific subclass of hyperholomorphic **m.v.d.f.** And later it will be explained in which meaning $\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z)$ is hyperholomorphic.

The equality (6.2.2) is not true, generally, for a function of class C^1; moreover the conditions (6.2.1), (6.2.2) are necessary and sufficient for g to be a holomorphic function in Ω^+.

These reasonings have their symmetric image for antiholomorphic functions.

6.3 A consequence for antiholomorphic functions

Let $G = gd\bar{z}E_{2\times 2}$ where g is an antiholomorphic function of several complex variables. Then G satisfies the formula (6.1.1), and combin-

ing it with Subsection 2.10, we have

$$g(z) = (-1)^{\frac{m(m-1)}{2}} \int_{\Gamma} \overline{\mathcal{U}}(\tau,\ z)\, g(\tau)\,, \tag{6.3.1}$$

$$0 = W_{jk}[g](z) - W_{kj}[g](z)\,, \tag{6.3.2}$$

where $j,\ k = 1,\ \ldots,\ m$, $j < k$, and the definitions of $W_{jk}[g](z)$, $W_{kj}[g](z)$ are in Subsection 2.11.8.

Note that the equality (6.3.1) is the Bochner-Martinelli integral representation of the antiholomorphic function g in the same domain Ω^+. Again (6.3.1) is the restriction of (6.1.1) onto a specific subclass of hyperholomorphic **m.v.d.f.,** which explains why a non-antiholomorphic kernel $\overline{\mathcal{U}}$ reproduces antiholomorphic functions. The conditions (6.3.1), (6.3.2) are necessary and sufficient for g to be an antiholomorphic function in Ω^+.

And now the general case will be considered.

6.4 A consequence for holomorphic-like functions

Let $G = g d\bar{z}^{\mathbf{j}} E_{2\times 2}$ where g is a complex function, that is antiholomorphic in variables $z_{j_1}, \ldots, z_{j_{|\mathbf{j}|}}$ and holomorphic in the rest, where $\mathbf{j} := \left(j_1,\ \ldots,\ j_{|\mathbf{j}|}\right)$ is strictly increasing $|\mathbf{j}|$-tuple in $\{1,\ \ldots,\ m\}$, with $|\mathbf{j}| = 1,\ \ldots,\ m-1$. Then G satisfies the formula (6.1.1) and combining it with (2.10), we have

$$g(z) = (-1)^{\frac{m(m-1)}{2}} \int_{\Gamma} \overline{\mathcal{U}}_{\mathbf{j}}(\tau,\ z)\, g(\tau)\,, \tag{6.4.1}$$

$$0 = U_{j_p k_q}[g](z) - \overline{U}_{k_q j_p}[g](z)\,, \tag{6.4.2}$$

where $k_q = 1,\ \ldots,\ m$ with $j_q < k_q < j_{q+1}$ for any $q = 0,\ \ldots,\ |\mathbf{j}|$, by definition $j_0 := 0$ and $j_{|\mathbf{j}|+1} := m+1$ and for $p = q$, $\frac{(p-q)-|p-q|}{2(p-q)} := 0$. For $|\mathbf{j}| = 1,\ \ldots,\ m-2$, we have

$$0 = \overline{W}_{jk}[g](z) - \overline{W}_{kj}[g](z)\,, \tag{6.4.3}$$

where $j,\ k = 1,\ \ldots,\ m$, $j,\ k \neq j_1,\ \ldots,\ j_{|\mathbf{j}|}$ and $j < k$. And, for $|\mathbf{j}| = 2,\ \ldots,\ m-1$, we have

$$0 = W_{j_p j_q}[g](z) - W_{j_q j_p}[g](z)\,, \tag{6.4.4}$$

where $p,\ q = 1,\ \ldots,\ |\mathbf{j}|$ and $p < q$.

The definitions of $U_{j_p k_q}$, $\overline{U}_{k_q j_p}$, $\overline{W}_{jk}$, $\overline{W}_{kj}$, $W_{j_p j_q}$ and $W_{j_q j_p}$ are in Subsection 2.11.8.

The equality (6.4.1) is the Bochner-Martinelli integral representation of a function antiholomorphic in the variables $z_{j_1},\ \ldots,\ z_{j_{|\mathbf{j}|}}$ and holomorphic in the rest in the same domain Ω^+. The conditions (6.4.1), (6.4.2), (6.4.3) and (6.4.4) are necessary and sufficient for g to be a function holomorphic in given variables and antiholomorphic in the rest.

We resume the above in the following corollaries.

6.5 Bochner-Martinelli integral representation for holomorphic functions of several complex variables, and hyperholomorphic function theory

Corollary *Let g be a function holomorphic in the bounded domain Ω^+, which can be extended continuously until the topological boundary Γ of Ω^+ which is a piecewise smooth surface. Then the hyperholomorphic Cauchy integral representation yields*

$$g(z) \;=\; (-1)^{\frac{m(m-1)}{2}} \int\limits_{\Gamma} \mathcal{U}(\tau,\ z)\, g(\tau)\,, \qquad z \in \Omega^+,$$

and besides

$$0 \;=\; \overline{W}_{jk}[g](z) - \overline{W}_{kj}[g](z)\,, \qquad z \in \Omega^+,$$

for any $j,\ k = 1,\ \ldots,\ m,\ j < k$.

6.6 Bochner-Martinelli integral representation for antiholomorphic functions of several complex variables, and hyperholomorphic function theory

Corollary *Let g be a function antiholomorphic in the bounded domain*

Ω^+, *which can be extended continuously until the topological boundary* Γ *of* Ω^+ *which is a piecewise smooth surface. Then the hyperholomorphic Cauchy integral representation yields*

$$g(z) = (-1)^{\frac{m(m-1)}{2}} \int_{\Gamma} \overline{\mathcal{U}}(\tau, z)\, g(\tau), \qquad z \in \Omega^+, \quad (6.6.1)$$

and besides

$$0 = W_{jk}[g](z) - W_{kj}[g](z), \qquad z \in \Omega^+, \qquad (6.6.2)$$

where $j, k = 1, \ldots, m$, $j < k$.

6.7 Bochner-Martinelli integral representation for functions holomorphic in some variables and antiholomorphic in the rest, and hyperholomorphic function theory

Corollary *Let* g *be a function antiholomorphic in the bounded domain* Ω^+ *in the variables* $z_{j_1}, \ldots, z_{j_{|\mathbf{j}|}}$, *and holomorphic in the rest, which can be extended continuously until the topological boundary* Γ *of* Ω^+, *which is a piecewise smooth surface, where* $\mathbf{j} := (j_1, \ldots, j_{|\mathbf{j}|})$ *is strictly increasing* $|\mathbf{j}|$*-tuple in* $\{1, \ldots, m\}$, *with* $|\mathbf{j}| = 1, \ldots, m-1$. *Then the hyperholomorphic Cauchy integral representation yields*

$$g(z) = (-1)^{\frac{m(m-1)}{2}} \int_{\Gamma} \overline{\mathcal{U}}_{\mathbf{j}}(\tau, z)\, g(\tau), \qquad (6.7.1)$$

and besides

$$0 = \left[U_{j_p k_q}[g](z) - \overline{U}_{k_q j_p}[g](z)\right], \qquad (6.7.2)$$

where $k_q = 1, \ldots, m$ *with* $j_q < k_q < j_{q+1}$ *for any* $q = 0, \ldots, |\mathbf{j}|$, *by definition* $j_0 := 0$ *and* $j_{|\mathbf{j}|+1} := m+1$ *and for* $p = q$, $\frac{(p-q)-|p-q|}{2(p-q)} := 0$. *For* $|\mathbf{j}| = 1, \ldots, m-2$, *we have also*

$$0 = \overline{W}_{jk}[g](z) - \overline{W}_{kj}[g](z), \qquad (6.7.3)$$

where $j,\ k = 1,\ \ldots,\ m$, $j,\ k \neq j_1,\ \ldots,\ j_{|\mathbf{j}|}$ *and* $j < k$. *And, for* $|\mathbf{j}| = 2,\ \ldots,\ m-1$, *we have also*

$$0 = W_{j_p j_q}[g](z) - W_{j_q j_p}[g](z), \tag{6.7.4}$$

where $p,\ q = 1,\ \ldots,\ |\mathbf{j}|$ *and* $p < q$.

Chapter 7

Hyperholomorphic $\overline{\mathcal{D}}$-problem

7.1 Some reasonings from one variable theory

Introduce the following integrals:

$$\begin{aligned} K_{\overline{\mathcal{D}}}[F](z) &= \int_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}), \\ K_{\mathcal{D}}[G](z) &= \int_{\Gamma} \mathcal{K}_{\mathcal{D}}(\tau - z,\ z) \underset{\star}{\wedge} \boldsymbol{\sigma}_{\tau,\, z} \underset{\star}{\wedge} G(\tau,\ dz) \end{aligned} \tag{7.1.1}$$

and

$$\begin{aligned} T_{\overline{\mathcal{D}}}[F](z) &= -\int_{\Omega^+} \mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} F(\tau,\ d\bar{z})\, dV_\tau, \\ T_{\mathcal{D}}[G](z) &= -\int_{\Omega^+} \mathcal{K}_{\mathcal{D}}(\tau - z,\ z) \underset{\star}{\wedge} G(\tau,\ dz)\, dV_\tau. \end{aligned} \tag{7.1.2}$$

Hence the Borel-Pompeiu formulas take the form

$$2F(z) = K_{\overline{\mathcal{D}}}[F](z) + T_{\overline{\mathcal{D}}} \circ \overline{\mathcal{D}}[F](z),$$

$$2G(z) = K_{\mathcal{D}}[G](z) + T_{\mathcal{D}} \circ \mathcal{D}[G](z). \tag{7.1.3}$$

For functions of one complex variable $\overline{\mathcal{D}} = \frac{\partial}{\partial \bar{z}}$ and

$$K_{\frac{\partial}{\partial \bar{z}}}[f](z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)\, d\tau}{\tau - z}, \tag{7.1.4}$$

$$T_{\frac{\partial}{\partial \bar{z}}}[f](z) = -\frac{1}{2\pi i} \int_{\Omega} \frac{f(\tau)}{\tau - z} d\overline{\tau} \wedge d\tau, \tag{7.1.5}$$

with the Borel-Pompeiu formula

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)\, d\tau}{\tau - z} - \frac{1}{2\pi i} \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{\tau}}(\tau)}{\tau - z} d\overline{\tau} \wedge d\tau, \tag{7.1.6}$$

which implies, as one of its multiple important consequences, the right-hand-side invertibility of the complex Cauchy-Riemann operator:

$$\frac{\partial}{\partial \overline{z}} \circ T_{\frac{\partial}{\partial \bar{z}}}[f] = f. \tag{7.1.7}$$

The latter leads immediately to the complete description of the set of solutions of the inhomogeneous (complex) Cauchy-Riemann equation: if

$$\frac{\partial f}{\partial \overline{z}} = g \tag{7.1.8}$$

in a domain Ω with smooth enough boundary $\Gamma = \partial\Gamma$, then

$$f \in \left(Hol(\Omega) + T_{\frac{\partial}{\partial \overline{z}}}[g] \right). \tag{7.1.9}$$

In the case of more than one variable, there is an analogue of (7.1.6) with the Bochner-Martinelli kernel now playing the role of the Cauchy kernel. But because of the non-holomorphy of the Bochner-Martinelli kernel, the properties of both summands are essentially different and do not imply an analogue of (7.1.7), which means that the $\overline{\partial}$-problem in several complex variables

$$\frac{\partial f}{\partial \overline{z}_1} = g_1,$$

$$\begin{aligned} & \cdots \\ \frac{\partial f}{\partial \overline{z}_m} &= g_m, \end{aligned} \tag{7.1.10}$$

is much more complicated; it does not have a simple formula such as (7.1.9) describing its solutions. The aim of this chapter is to show that in the hyperholomorphic setting the hyperholomorphic Cauchy-Riemann operator does have the right-hand-side inverse, and hence the inhomogeneous hyperholomorphic Cauchy-Riemann equation allows an easy description of the same form as (7.1.9). What is more, having as a base the solution of the hyperholomorphic $\overline{\mathcal{D}}$-problem, in Chapter 8 we obtain the necessary and sufficient conditions for the solvability of the $\overline{\partial}$-problem for complex-valued functions, together with the explicit formula for the solution itself when it exists. In the case of the $\overline{\partial}$-problem for differential forms, the necessary and sufficient conditions obtained are not so efficient, but the formula for the solution here is also quite explicit.

7.2 Right inverse operators to the hyperholomorphic Cauchy-Riemann operators

Theorem *Let Ω^+ be a bounded open set in $\mathbb{C}^m$ with the topological boundary Γ, which is a piecewise smooth surface. In the spaces $\overline{\mathfrak{G}}_1(\Omega^+) \cap \overline{\mathfrak{G}}_0(\Omega^+ \cup \Gamma)$ and $\mathfrak{G}_1(\Omega^+) \cap \mathfrak{G}_0(\Omega^+ \cup \Gamma)$, the operators $\overline{\mathcal{D}} \circ T_{\overline{\mathcal{D}}}$ and $\mathcal{D} \circ T_{\mathcal{D}}$ are well-defined and, respectively, on these sets:*

$$\begin{aligned} \overline{\mathcal{D}} \circ T_{\overline{\mathcal{D}}}[F] &= 2F, \\ \mathcal{D} \circ T_{\mathcal{D}}[G] &= 2G. \end{aligned}$$

Proof. Assume that F belongs to $\overline{\mathfrak{G}}_2(\Omega^+) \cap \overline{\mathfrak{G}}_0(\Omega^+ \cup \Gamma)$ (or to some other appropriate dense subset of $\overline{\mathfrak{G}}_1(\Omega^+) \cap \overline{\mathfrak{G}}_0(\Omega^+ \cup \Gamma)$). Take $\theta_{\mathbb{C}^m}$ to be a fundamental solution of the complex Laplace operator on $\mathbb{C}^m$ (see Subsection 1.3), i.e., for $z \neq 0$,

$$\theta_{\mathbb{C}^m}(z) := \begin{cases} -2\frac{(m-2)!}{\pi^m}\frac{1}{|z|^{2(m-1)}}, & m > 1, \\ \\ \frac{2}{\pi}\ln(|z|), & m = 1; \end{cases}$$

and

$$\Delta_{\mathbb{C}^m}\left[\theta_{\mathbb{C}^m}\right] = \delta_{\mathbb{C}^m}$$

(Dirac's delta on $\mathbb{C}^m$) in the distributional sense. From Theorem 2.7 we have that for $\tau \neq z$,

$$\begin{aligned} d_\tau \left(\theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) \right) = \\ = \left(\overline{\mathcal{D}}^*_r \left[\theta_{\mathbb{C}^m} E_{2\times 2}\right](\tau - z) \underset{\star}{\wedge} F(\tau,\ d\bar{z}) + \right. \\ \left. + \theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} \overline{\mathcal{D}}^*\left[F\right](\tau,\ d\bar{z}) \right) dV_\tau, \end{aligned}$$

or, in the integral form,

$$\int\limits_{\Gamma} \theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}) =$$

$$\begin{aligned} = \int\limits_{\Omega^+} \mathcal{K}_{\overline{\mathcal{D}}}(\tau,\ z) \underset{\star}{\wedge} F(\tau,\ d\bar{z})\, dV_\tau + \\ + \int\limits_{\Omega^+} \theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} \overline{\mathcal{D}}^*\left[F\right](\tau,\ d\bar{z})\, dV_\tau, \end{aligned}$$

which implies

$$\begin{aligned} T_{\overline{\mathcal{D}}}[F] = \int\limits_{\Omega^+} \theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} \overline{\mathcal{D}}^*\left[F\right](\tau,\ d\bar{z})\, dV_\tau - \\ - \int\limits_{\Gamma} \theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\ d\bar{z}). \end{aligned}$$

Apply operator $\overline{\mathcal{D}}$ to both sides of the last equality. It is straightforward to show that for $\theta_{\mathbb{C}^m}(\tau - z)$ and for any $H(\tau,\, d\tau,\, d\bar{\tau},\, d\bar{z})$ we have

$$\begin{aligned} \overline{\mathcal{D}}_z \left[\theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2} \underset{\star}{\wedge} H(\tau,\ d\tau,\ d\bar{\tau},\ d\bar{z}) \right] = \\ = \overline{\mathcal{D}}_{r;\, z} \left[\theta_{\mathbb{C}^m}(\tau - z)\, E_{2\times 2}\right] \underset{\star}{\wedge} H(\tau,\ d\tau,\ d\bar{\tau},\ d\bar{z}). \end{aligned}$$

Hence

$$
\begin{aligned}
\overline{\mathcal{D}} \circ T_{\overline{\mathcal{D}}}[F] &= \int\limits_{\Omega^+} \overline{\mathcal{D}}_{r;\,z}\left[\theta_{\mathbb{C}^m}\,(\tau - z)\,E_{2\times 2}\right] \underset{\star}{\wedge} \overline{\mathcal{D}}^*\,[F]\,(\tau,\ d\bar{z})\,dV_\tau - \\
&\quad - \int\limits_{\Gamma} \overline{\mathcal{D}}_{r;\,z}\left[\theta_{\mathbb{C}^m}\,(\tau - z)\,E_{2\times 2}\right] \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau,\,z} \underset{\star}{\wedge} \\
&\qquad \underset{\star}{\wedge} F\,(\tau,\ d\bar{z}) = \\
&= \int\limits_{\Gamma} \overline{\mathcal{D}}_{r;\,\tau}\left[\theta_{\mathbb{C}^m}\,(\tau - z)\,E_{2\times 2}\right] \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau,\,z} \underset{\star}{\wedge} F\,(\tau,\ d\bar{z}) - \\
&\quad - \int\limits_{\Omega^+} \overline{\mathcal{D}}_{r;\,\tau}\left[\theta_{\mathbb{C}^m}\,(\tau - z)\,E_{2\times 2}\right] \underset{\star}{\wedge} \\
&\qquad \underset{\star}{\wedge} \overline{\mathcal{D}}^*\,[F]\,(\tau,\ d\bar{z})\,dV_\tau = \\
&= \int\limits_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}^*}\,(\tau,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau,\,z} \underset{\star}{\wedge} F\,(\tau,\ d\bar{z}) - \\
&\quad - \int\limits_{\Omega^+} \mathcal{K}_{\overline{\mathcal{D}}^*}\,(\tau,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}^*\,[F]\,(\tau,\ d\bar{z})\,dV_\tau = \\
&= K_{\overline{\mathcal{D}}^*}\,[F] + T_{\overline{\mathcal{D}}^*} \circ \overline{\mathcal{D}}^*\,[F] = 2F.
\end{aligned}
$$

On the last step we made use of the Borel-Pompeiu formula consistent with the operator $\overline{\mathcal{D}}^*$, which can be proved directly as in Subsection 2.11 or can be obtained from the formula (2.11.1) using the relation (2.3.8). ■

7.2.1 Structure of the formula of Theorem 7.2

Theorem 7.2 has useful consequences both in hyperholomorphic terms and in purely complex analysis terms. We start with the latter. To this end, the equality

$$\overline{\mathcal{D}} \circ T_{\overline{\mathcal{D}}}[F] = 2F$$

is equivalent to

$$
\begin{pmatrix}
\bar{\partial}[B^{11}] + \bar{\partial}^*[B^{21}]; & \bar{\partial}[B^{12}] + \bar{\partial}^*[B^{22}] \\
\bar{\partial}^*[B^{11}] + \bar{\partial}[B^{21}]; & \bar{\partial}^*[B^{12}] + \bar{\partial}[B^{22}]
\end{pmatrix} = 2F,
$$

where

$$
\begin{aligned}
B^{11}(z) &= -2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\int\limits_{\Omega^+}\left(\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge F^{11}(\tau,\ d\bar z)+\right.\\
&\quad\left.+\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge F^{21}(\tau,\ d\bar z)\right)dV_\tau,\\
B^{21}(z) &= -2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\int\limits_{\Omega^+}\left(\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge F^{11}(\tau,\ d\bar z)+\right.\\
&\quad\left.+\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge F^{21}(\tau,\ d\bar z)\right)dV_\tau,\\
B^{12}(z) &= -2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\int\limits_{\Omega^+}\left(\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge F^{12}(\tau,\ d\bar z)+\right.\\
&\quad\left.+\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge F^{22}(\tau,\ d\bar z)\right)dV_\tau,\\
B^{22}(z) &= -2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\int\limits_{\Omega^+}\left(\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge F^{12}(\tau,\ d\bar z)+\right.\\
&\quad\left.+\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge F^{22}(\tau,\ d\bar z)\right)dV_\tau.
\end{aligned}
$$

Hence for all entries $F^{\alpha\beta}(z)$ of the matrix $F(z)$ we have

$$
\begin{aligned}
&2F^{\alpha\beta}(z)=\\
&\quad=\ \bar\partial\left[-2\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\sum_{q=1}^{m}\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge F^{\alpha\beta}(\tau,\ d\bar z)\,dV_\tau\right]+\\
&\qquad+\bar\partial^*\left[-2\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge F^{\alpha\beta}(\tau,\ d\bar z)\,dV_\tau\right],
\end{aligned}
\tag{7.2.1}
$$

$$0 = \bar{\partial}^* \left[2\frac{(m-1)!}{\pi^m} \int\limits_{\Omega^+} \sum_{q=1}^{m} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q} \wedge F^{\alpha\beta}(\tau,\ d\bar{z})\, dV_\tau \right] +$$

$$+ \bar{\partial} \left[2\frac{(m-1)!}{\pi^m} \int\limits_{\Omega^+} \sum_{q=1}^{m} \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q \wedge F^{\alpha\beta}(\tau,\ d\bar{z})\, dV_\tau \right]. \tag{7.2.2}$$

The equality (7.2.1) can be seen as an integral representation of a smooth differential form $F^{\alpha\beta}$ with differentiations outside the integrals.

Consider now what (7.2.1) and (7.2.2) give in different dimensions.

7.2.2 Case $m = 1$

There are differential forms of degree 1 and of degree 0 here. First let $F = \begin{pmatrix} \psi_1^{11} & \psi_1^{12} \\ \psi_1^{21} & \psi_1^{22} \end{pmatrix} d\bar{z} = F_1 d\bar{z}$; then we have from (7.2.1)

$$F_1(z) = \frac{\partial}{\partial \bar{z}} \left[-\frac{1}{\pi} \int\limits_{\Omega^+} \frac{F_1(\tau)}{\tau - z} dV_\tau \right],$$

or, using a commonly accepted notation $T[F_1]$ for the above integral,

$$F_1 = \frac{\partial}{\partial \bar{z}} T[F_1],$$

a well-known formula from one-dimensional complex analysis, which expresses the existence of a right inverse operator to the complex Cauchy-Riemann operator, a fact of enormous importance for the whole theory in one variable. Of course Theorem 7.2 shows a complete analogy with the hyperholomorphic situation, unlike that of several complex variables.

The equality (7.2.2) gives no new information, since both summands in the right-hand side are identically zero.

Let now $F = \begin{pmatrix} \varphi^{11} & \varphi^{12} \\ \varphi^{21} & \varphi^{22} \end{pmatrix}$; so we have from (7.2.1)

$$\begin{aligned} F(z) &= \frac{\partial}{\partial z}\left[-\frac{1}{\pi}\int\limits_{\Omega^+}\frac{F(\tau)}{\bar{\tau}-\bar{z}}dV_\tau\right], \\ F &= \frac{\partial}{\partial z}\overline{T}[F]. \end{aligned}$$

This is an analogue of the above formula but for antiholomorphic functions, and all the comments can be repeated.

7.2.3 Case $m = 2$

Here the variety of options is greater, and the conclusions are different. We consider subsequently what (7.2.1) and (7.2.2) give for **m.v.d.f.** of a fixed degree. First, for the case of degree two, let $F = \begin{pmatrix} \psi_{12}^{11} & \psi_{12}^{12} \\ \\ \psi_{12}^{21} & \psi_{12}^{22} \end{pmatrix} d\bar{z} = F_{12}\, d\bar{z}$; so we have from (7.2.1)

$$\begin{aligned} F(z) &= \bar{\partial}\left[\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau d\bar{z}^1 - \right. \\ &\quad \left. -\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau d\bar{z}^2\right], \end{aligned}$$

and from (7.2.2)

$$\begin{aligned} 0 &= \bar{\partial}^*\left[\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau d\bar{z}^1 - \right. \\ &\quad \left. -\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau d\bar{z}^2\right]; \end{aligned}$$

hence

$$F_{12}(z) = \frac{\partial}{\partial\bar{z}^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau\right] +$$

$$+\frac{\partial}{\partial\bar{z}^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau\right],$$

$$\begin{aligned}0 &= -\frac{\partial}{\partial z_1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau\right]+\\ &\quad+\frac{\partial}{\partial z_2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_{12}(\tau)\,dV_\tau\right].\end{aligned}$$

It is quite instructive, again, to look at these equalities written in the operator form. This will be done below for an arbitrary number of variables (see Theorems 7.2.5 and 7.2.6.)

Now we pass to **m.v.d.f.** of degree one.

Let $F=\begin{pmatrix}\psi_1^{11} & \psi_1^{12}\\ \psi_1^{21} & \psi_1^{22}\end{pmatrix}d\bar{z}^1=F_1d\bar{z}^1$; so we have from (7.2.1)

$$\begin{aligned}F_1(z)\,d\bar{z}^1 &= \bar{\partial}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]-\\ &\quad-\bar{\partial}^*\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F_1(\tau)\,dV_\tau d\bar{z}^1\wedge d\bar{z}^2\right]=\\ &= \frac{\partial}{\partial\bar{z}^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]d\bar{z}^1+\\ &\quad+\frac{\partial}{\partial\bar{z}^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]d\bar{z}^2-\\ &\quad-\frac{\partial}{\partial z^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]d\bar{z}^2+\\ &\quad+\frac{\partial}{\partial z^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]d\bar{z}^1,\end{aligned}$$

and we obtain nothing new from (7.2.2), since both summands in the right-hand-side are identically zero. Hence we get two equalities for a given smooth matrix F_1:

$$\begin{aligned} F_1(z) &= \frac{\partial}{\partial \bar{z}^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]+ \\ &\quad +\frac{\partial}{\partial z^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right], \\ 0 &= \frac{\partial}{\partial \bar{z}^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_1-\bar{z}_1}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]- \\ &\quad -\frac{\partial}{\partial z^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F_1(\tau)\,dV_\tau\right]. \end{aligned}$$

Analogously, let $F=\begin{pmatrix}\psi_2^{11} & \psi_2^{12}\\ \psi_2^{21} & \psi_2^{22}\end{pmatrix}d\bar{z}^2=F_2 d\bar{z}^2$; so we have from (7.2.1)

$$\begin{aligned} F_2(z)\,d\bar{z}^2 &= \bar{\partial}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]+ \\ &\quad +\bar{\partial}^*\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F_2(\tau)\,dV_\tau d\bar{z}^1\wedge d\bar{z}^2\right]= \\ &= \frac{\partial}{\partial \bar{z}^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]d\bar{z}^1+ \\ &\quad +\frac{\partial}{\partial \bar{z}^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]d\bar{z}^2+ \\ &\quad +\frac{\partial}{\partial z^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]d\bar{z}^2- \end{aligned}$$

$$-\frac{\partial}{\partial z^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]d\bar{z}^1,$$

and we have nothing new from (7.2.2). Hence, for a given smooth matrix F_2, we get two equalities symmetric, with respect to the variables z_1 and z_2, to those for F_1:

$$\begin{aligned}
F_2(z) &= \frac{\partial}{\partial \bar{z}^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]+\\
&\quad+\frac{\partial}{\partial z^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right],\\
0 &= \frac{\partial}{\partial \bar{z}^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\bar{\tau}_2-\bar{z}_2}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right]-\\
&\quad-\frac{\partial}{\partial z^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F_2(\tau)\,dV_\tau\right].
\end{aligned}$$

Finally we consider **m.v.d.f.** of degree zero, for which let

$$F=\begin{pmatrix}\varphi^{11} & \varphi^{12}\\ \varphi^{21} & \varphi^{22}\end{pmatrix};$$

so we have from (7.2.1)

$$\begin{aligned}
F(z) &= \bar{\partial}^*\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F(\tau)\,dV_\tau d\bar{z}^1+\right.\\
&\quad\left.+\left(-\frac{1}{\pi^2}\right)\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F(\tau)\,dV_\tau d\bar{z}^2\right];
\end{aligned}$$

and from (7.2.2)

$$0 = \bar{\partial}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F(\tau)\,dV_\tau d\bar{z}^1+\right.$$

$$+\left(-\frac{1}{\pi^2}\right)\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F(\tau)\,dV_\tau d\bar{z}^2\Bigg].$$

Hence for a given smooth matrix F we get two equalities:

$$\begin{aligned}F(z) &= \frac{\partial}{\partial z^1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F(\tau)\,dV_\tau\right]+\\ &\quad+\frac{\partial}{\partial z^2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F(\tau)\,dV_\tau\right],\end{aligned}$$

$$\begin{aligned}0 &= \frac{\partial}{\partial z_1}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_2-z_2}{|\tau-z|^4}F(\tau)\,dV_\tau\right]-\\ &\quad-\frac{\partial}{\partial z_2}\left[-\frac{1}{\pi^2}\int\limits_{\Omega^+}\frac{\tau_1-z_1}{|\tau-z|^4}F(\tau)\,dV_\tau\right].\end{aligned}$$

Because all participating operators are linear, the case of an arbitrary form does not give any new formula. Operator interpretations of all this will be given later on.

7.2.4 Case $m>2$

The quite detailed cases of one and two variables considered above give a good idea of what may occur in general situations; thus, we shall present it now fluently, without too many explanations.

Let $F=\begin{pmatrix}\psi_{\mathbf{j}}^{11} & \psi_{\mathbf{j}}^{12}\\ \\ \psi_{\mathbf{j}}^{21} & \psi_{\mathbf{j}}^{22}\end{pmatrix}d\bar{z}^{\mathbf{j}}=F_{\mathbf{j}}d\bar{z}^{\mathbf{j}}$; so we have from (7.2.1)

$$\begin{aligned}&F_{\mathbf{j}}(z)\,d\bar{z}^{\mathbf{j}}=\\ &=\bar{\partial}\Bigg[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\sum_{n=j_1,\,\ldots,\,j_{|\mathbf{j}|}}\frac{\bar{\tau}_n-\bar{z}_n}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\cdot\\ &\qquad\cdot(-1)^{n-1}\,d\bar{z}^{j_1}\wedge\ldots\wedge d\bar{z}^{j_{n-1}}\wedge d\bar{z}^{j_{n+1}}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}\Bigg]+\end{aligned}$$

$$+\,\bar{\partial}^{*}\left[-\frac{(m-1)!}{\pi^{m}}\int\limits_{\Omega^{+}}\sum_{p=0}^{|\mathbf{j}|}\sum_{j_{p}<n<j_{p+1}}\frac{\tau_{n}-z_{n}}{|\tau-z|^{2m}}F_{\mathbf{j}}\left(\tau\right)dV_{\tau}\cdot\right.$$

$$\left.\cdot(-1)^{p}d\bar{z}^{j_{1}}\wedge\ldots\wedge d\bar{z}^{j_{p}}\wedge d\bar{z}^{n}\wedge d\bar{z}^{j_{p+1}}\wedge\cdots\wedge d\bar{z}^{j_{|\mathbf{j}|}}\right]=$$

$$=\sum_{n=j_{1},\,\ldots,\,j_{|\mathbf{j}|}}\sum_{q=0}^{j-2}\sum_{j_{q}<k<j_{q+1}}$$

$$\frac{\partial}{\partial\bar{z}_{k}}\left[-\frac{(m-1)!}{\pi^{m}}\int\limits_{\Omega^{+}}\frac{\bar{\tau}_{n}-\bar{z}_{n}}{|\tau-z|^{2m}}F_{\mathbf{j}}\left(\tau\right)dV_{\tau}\right]\cdot$$

$$\cdot\left(-1\right)^{q}\left(-1\right)^{n-1}d\bar{z}^{j_{1}}\wedge\ldots\wedge d\bar{z}^{j_{q}}\wedge d\bar{z}^{k}\wedge d\bar{z}^{j_{q+1}}\wedge\ldots$$

$$\ldots\wedge d\bar{z}^{j_{n-1}}\wedge d\bar{z}^{j_{n+1}}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}+$$

$$+\sum_{n=j_{1},\,\ldots,\,j_{|\mathbf{j}|}}\sum_{j_{n-1}<k<j_{n+1}}$$

$$\frac{\partial}{\partial\bar{z}_{k}}\left[-\frac{(m-1)!}{\pi^{m}}\int\limits_{\Omega^{+}}\frac{\bar{\tau}_{n}-\bar{z}_{n}}{|\tau-z|^{2m}}F_{\mathbf{j}}\left(\tau\right)dV_{\tau}\right]\cdot$$

$$\cdot\,d\bar{z}^{j_{1}}\wedge\ldots\wedge d\bar{z}^{j_{n-1}}\wedge d\bar{z}^{k}\wedge d\bar{z}^{j_{n+1}}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}+$$

$$+\sum_{n=j_{1},\,\ldots,\,j_{|\mathbf{j}|}}\sum_{q=n+1}^{m}\sum_{j_{q}<k<j_{q+1}}$$

$$\frac{\partial}{\partial\bar{z}_{k}}\left[-\frac{(m-1)!}{\pi^{m}}\int\limits_{\Omega^{+}}\frac{\bar{\tau}_{n}-\bar{z}_{n}}{|\tau-z|^{2m}}F_{\mathbf{j}}\left(\tau\right)dV_{\tau}\right]\cdot$$

$$\cdot\left(-1\right)^{q-1}\left(-1\right)^{n-1}d\bar{z}^{j_{1}}\wedge\ldots\wedge d\bar{z}^{j_{n-1}}\wedge d\bar{z}^{j_{n+1}}\wedge\ldots$$

$$\ldots\wedge d\bar{z}^{j_{q}}\wedge d\bar{z}^{k}\wedge d\bar{z}^{j_{q+1}}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}+$$

$$+\bar{\partial}^*\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\sum_{p=0}^{|\mathbf{j}|}\sum_{j_p<n<j_{p+1}}\frac{\tau_n-z_n}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)dV_\tau\cdot\right.$$

$$\left.\cdot(-1)^p\,d\bar{z}^{j_1}\wedge\ldots\wedge d\bar{z}^{j_p}\wedge d\bar{z}^{n}\wedge d\bar{z}^{j_{p+1}}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}\right],$$

and combining it with (7.2.2) we can conclude the following. Let $\mathbf{j}$ be a strictly increasing $|\mathbf{j}|$-tuple in $\{1,\ \ldots,\ m\}$; take an **arbitrary** smooth matrix-function which we denote by $F_{\mathbf{j}}(z)$ for the sake of convenience and to emphasize the presence of $\mathbf{j}$ in the formulas below. Then

$$\begin{aligned}F_{\mathbf{j}}(z) &= \sum_{k=1}^{|\mathbf{j}|}\frac{\partial}{\partial\bar{z}_{j_k}}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\bar{\tau}_{j_k}-\bar{z}_{j_k}}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right]+\\ &+\sum_{k=0}^{|\mathbf{j}|}\sum_{j_k<n<j_{k+1}}\frac{\partial}{\partial z_j}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\tau_n-z_n}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right];\end{aligned}$$

for $0<|\mathbf{j}|<m$, $k=1,\ \ldots,\ |\mathbf{j}|$, $n\neq j_1,\ \ldots,\ j_{|\mathbf{j}|}$, there holds

$$\begin{aligned}0 &= \frac{\partial}{\partial\bar{z}_j}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\bar{\tau}_{j_k}-\bar{z}_{j_k}}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right]-\\ &-\frac{\partial}{\partial z_{j_k}}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\tau_n-z_n}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right],\end{aligned}$$

for $1<|\mathbf{j}|\leq m$, $j_k\neq j_p$, there holds

$$\begin{aligned}0 &= \frac{\partial}{\partial z_{j_p}}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\bar{\tau}_{j_k}-\bar{z}_{j_k}}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right]-\\ &-\frac{\partial}{\partial z_{j_k}}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\bar{\tau}_{j_p}-\bar{z}_{j_p}}{|\tau-z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right],\end{aligned}$$

and finally for $0 \leq |\mathbf{j}| < m-1$; $q \neq n$; $q,\ n \neq j_1,\ \ldots,\ j_{|\mathbf{j}|}$ there holds

$$0 = \frac{\partial}{\partial \bar{z}_q}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\tau_n - z_n}{|\tau - z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right] - \\ -\frac{\partial}{\partial \bar{z}_n}\left[-\frac{(m-1)!}{\pi^m}\int\limits_{\Omega^+}\frac{\tau_q - z_q}{|\tau - z|^{2m}}F_{\mathbf{j}}(\tau)\,dV_\tau\right].$$

Now we resume the above computations in the following theorems, where notations from Subsection 2.11.8 are used.

7.2.5 Analogs of (7.1.7)

Theorem *Let $\Omega^+ \subset \mathbb{C}^m$ be a bounded domain with the piecewise smooth boundary. On the space $C^1(\Omega^+;\ \mathbb{C})$, for any $j = 1,\ \ldots,\ m$, the operators $\frac{\partial}{\partial \bar{z}_j} \circ T_j$ and $\frac{\partial}{\partial z_j} \circ \overline{T}_j$ are well defined and, on this set,*

$$f = \sum\limits_{\substack{j=1 \\ j \neq j_1,\ \ldots,\ j_{|\mathbf{j}|}}}^{m} \frac{\partial}{\partial \bar{z}_j} \circ T_j[f] + \sum\limits_{j = j_1,\ \ldots,\ j_{|\mathbf{j}|}} \frac{\partial}{\partial z_j} \circ \overline{T}_j[f],$$

where $\mathbf{j}$ is a strictly increasing $|\mathbf{j}|$-tuple in $\{1,\ \ldots,\ m\}$.

In particular, we have

$$f = \sum\limits_{j=1}^{m} \frac{\partial}{\partial \bar{z}_j} \circ T_j[f],$$

and symmetrically

$$f = \sum\limits_{j=1}^{m} \frac{\partial}{\partial z_j} \circ \overline{T}_j[f].$$

For $m = 1$ we have the formulas from Subsection 7.2.2, which give a right-hand-side invertibility of the Cauchy-Riemann operators, while for $m > 1$ the situation is radically different in principle.

7.2.6 Commutativity relations for T-type operators

Theorem *Let $\Omega^+ \subset \mathbb{C}^m$ be a bounded domain with the piecewise smooth boundary. On the space $C^1(\Omega^+;\ \mathbb{C})$, for any $j,\ k = 1,\ \ldots,\ m,\ j \neq k$, the operators $\frac{\partial}{\partial \bar{z}_j} \circ T_k$, $\frac{\partial}{\partial z_j} \circ \overline{T}_k$, $\frac{\partial}{\partial \bar{z}_j} \circ \overline{T}_k$ and $\frac{\partial}{\partial z_j} \circ T_k$ are well defined and, on this set,*

$$\begin{aligned} 0 &= \frac{\partial}{\partial \bar{z}_j} \circ T_k[f] - \frac{\partial}{\partial z_k} \circ \overline{T}_j[f], \\ 0 &= \frac{\partial}{\partial z_j} \circ T_k[f] - \frac{\partial}{\partial z_k} \circ T_j[f]. \end{aligned}$$

Note that the last equality is equivalent to

$$0 = \frac{\partial}{\partial \bar{z}_j} \circ \overline{T}_k[f] - \frac{\partial}{\partial \bar{z}_k} \circ \overline{T}_j[f].$$

The last theorem gives a kind of "commutativity relations" for subindices of partial derivatives $\frac{\partial}{\partial \bar{z}_j}$ and $\frac{\partial}{\partial z_k}$, on one hand, and of operators $\overline{T}_j$ and T_k, on the other.

7.3 Solution of the hyperholomorphic $\overline{\mathcal{D}}$-problem

Theorem *Let Ω^+ be a bounded open set in $\mathbb{C}^m$ with the topological boundary Γ, which is a piecewise smooth surface. Consider the following equation:*

$$\overline{\mathcal{D}}[F] = G, \tag{7.3.1}$$

where $G \in \overline{\mathfrak{G}}_1(\Omega^+) \cap \overline{\mathfrak{G}}_0(\Omega^+ \cup \Gamma)$. Each solution of the above equation is of the form

$$F = \frac{1}{2} T_{\overline{\mathcal{D}}}[G] + H,$$

where H is an arbitrary element of $\overline{\mathfrak{N}}(\Omega^+)$.

Theorem 7.3 is a direct corollary of Theorem 7.2.

7.4 Structure of the general solution of the hyperholomorphic $\overline{\mathcal{D}}$-problem

Explicitly, the equality (7.3.1) has the form

$$\begin{aligned} \bar{\partial}\left[F^{11}\right] + \bar{\partial}^{*}\left[F^{21}\right] &= G^{11}, \\ \bar{\partial}^{*}\left[F^{11}\right] + \bar{\partial}\left[F^{21}\right] &= G^{21}, \end{aligned} \tag{7.4.1}$$

$$\begin{aligned} \bar{\partial}\left[F^{12}\right] + \bar{\partial}^{*}\left[F^{22}\right] &= G^{12}, \\ \bar{\partial}^{*}\left[F^{12}\right] + \bar{\partial}\left[F^{22}\right] &= G^{22}, \end{aligned} \tag{7.4.2}$$

where G^{11}, G^{21}, G^{12} and G^{22} are arbitrary elements of $\overline{\mathcal{G}}_1(\Omega^+) \cap \overline{\mathcal{G}}_0(\Omega^+ \cup \Gamma)$.

Note that the systems (7.4.1) and (7.4.2) are independent; thus, we shall study the system (7.4.1) only.

Each solution of the system (7.4.1) can be written as a (2×2)-matrix-valued differential form $\begin{pmatrix} F^{11} & 0 \\ F^{21} & 0 \end{pmatrix}$, which is a solution of the $\overline{\mathcal{D}}$-problem:

$$\overline{\mathcal{D}}[F] = \begin{pmatrix} G^{11} & 0 \\ G^{21} & 0 \end{pmatrix}. \tag{7.4.3}$$

To obtain the general solution of the system (7.4.1), it is sufficient to look at the general solution of the $\overline{\mathcal{D}}$-problem (7.4.3), which is in Theorem 7.3.

Each solution of the $\overline{\mathcal{D}}$-problem (7.4.3) is of the form

$$F = \frac{1}{2} T_{\overline{\mathcal{D}}}\left[\begin{pmatrix} G^{11} & 0 \\ G^{21} & 0 \end{pmatrix}\right] + H,$$

where H is an arbitrary element of $\overline{\mathfrak{N}}(\Omega^+)$. But, by definition,

$$\frac{1}{2} T_{\overline{\mathcal{D}}}\left[\begin{pmatrix} G^{11} & 0 \\ G^{21} & 0 \end{pmatrix}\right] :=$$

$$:= -\int_{\Omega^+} \frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \begin{pmatrix} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q}; & \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q \\ \\ \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q; & \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q} \end{pmatrix} \overset{\wedge}{\star}$$

$$\overset{\wedge}{\star} \begin{pmatrix} G^{11}(\tau,\ d\bar{z}); & 0 \\ G^{21}(\tau,\ d\bar{z}); & 0 \end{pmatrix} dV_\tau.$$

Hence, for $G^{11} := \sum\limits_{\mathbf{j}} G^{11}_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ and $G^{21} := \sum\limits_{\mathbf{k}} G^{21}_{\mathbf{k}} d\bar{z}^{\mathbf{k}}$, there holds

$$\frac{1}{2} T_{\overline{\mathcal{D}}} \left[\begin{pmatrix} G^{11}; & 0 \\ G^{21}; & 0 \end{pmatrix} \right] = \begin{pmatrix} N^{11}; & 0 \\ N^{21}; & 0 \end{pmatrix},$$

where

$$\begin{aligned} N^{11} \quad := \quad & \sum_{\mathbf{j}} \sum_{q=1}^{|\mathbf{j}|} (-1)^{q-1} T_{j_q} \left[G^{11}_{\mathbf{j}} \right] d\bar{z}^{j_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{j_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ & + \sum_{\mathbf{k}} \sum_{p=1}^{|\mathbf{k}|+1} \sum_{q=k_{p-1}+1}^{k_p - 1} (-1)^{p-1} \bar{T}_q \left[G^{21}_{\mathbf{k}} \right] d\bar{z}^{k_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{k_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{k_p} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}}; \end{aligned}$$

$$\begin{aligned} N^{21} \quad := \quad & \sum_{\mathbf{j}} \sum_{p=1}^{|\mathbf{j}|+1} \sum_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \bar{T}_q \left[G^{11}_{\mathbf{j}} \right] d\bar{z}^{j_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ & + \sum_{\mathbf{k}} \sum_{q=1}^{|\mathbf{k}|} (-1)^{q-1} T_{k_q} \left[G^{21}_{\mathbf{k}} \right] d\bar{z}^{k_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{k_{q-1}} \wedge d\bar{z}^{k_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}}. \end{aligned}$$

Therefore, each solution of the system (7.4.3) is of the form

$$F \;=\; \begin{pmatrix} F^{11}; & 0 \\ F^{21}; & 0 \end{pmatrix}$$

where

$$\begin{aligned} F^{11} \;:=\; & \sum_{\mathbf{j}} \sum_{q=1}^{|\mathbf{j}|} (-1)^{q-1} T_{j_q} \left[G^{11}_{\mathbf{j}} \right] d\bar{z}^{j_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{j_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ & + \sum_{|\mathbf{k}|} \sum_{p=1}^{k+1} \sum_{q=k_{p-1}+1}^{k_p - 1} (-1)^{p-1} \bar{T}_q \left[G^{21}_{\mathbf{k}} \right] d\bar{z}^{k_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{k_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{k_p} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} \;+\; H^{11}, \end{aligned}$$

$$\begin{aligned} F^{21} \;:=\; & \sum_{\mathbf{j}} \sum_{p=1}^{|\mathbf{j}|+1} \sum_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1} \bar{T}_q \left[G^{11}_{\mathbf{j}} \right] d\bar{z}^{j_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ & + \sum_{\mathbf{k}} \sum_{q=1}^{|\mathbf{k}|} (-1)^{q-1} T_{k_q} \left[G^{21}_{\mathbf{k}} \right] d\bar{z}^{k_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{k_{q-1}} \wedge d\bar{z}^{k_{q+1}} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} \;+\; H^{21}, \end{aligned} \tag{7.4.4}$$

where H^{11} and H^{21} satisfy the following conditions:

$$\begin{aligned} \bar{\partial} \left[H^{11} \right] + \bar{\partial}^{*} \left[H^{21} \right] &= 0, \\ \bar{\partial}^{*} \left[H^{11} \right] + \bar{\partial} \left[H^{21} \right] &= 0, \end{aligned}$$

this is

$$\begin{pmatrix} H^{11} & 0 \\ H^{21} & 0 \end{pmatrix} \in \overline{\mathfrak{N}} \left(\Omega^{+} \right).$$

7.5 $\overline{\mathcal{D}}$-type problem for the Hodge-Dirac operator

As a matter of fact, the equation (7.3.1) contains many $\overline{\partial}$-*type problems*. To illustrate this idea, which will be further elaborated on in the next chapter, consider the $\overline{\mathcal{D}}$-type problem for the Hodge-Dirac operator, (see Section 9.3):

$$\left(\bar{\partial}+\bar{\partial}^{*}\right)[f]=g, \tag{7.5.1}$$

where g is an arbitrary element of $\overline{\mathcal{G}}_1\left(\Omega^{+}\right)\cap\overline{\mathcal{G}}_0\left(\Omega^{+}\cup\Gamma\right)$.

Each solution of the equation (7.5.1) can be written as a solution of the system (7.4.1) when $G^{11}:=G^{21}:=g$, and $F^{11}=F^{21}=f$. Hence, to obtain all solutions of the equation (7.5.1), it is sufficient to look at all solutions of the system (7.4.2) when $G^{11}:=G^{21}:=g$, which are in (7.4.4).

When $G^{11}:=G^{21}:=g:=\sum\limits_{\mathbf{j}}g_{\mathbf{j}}d\bar{z}^{\mathbf{j}}$, (7.4.4) takes the form

$$\begin{aligned}
F^{11} \;:=\; & \sum_{\mathbf{j}}\sum_{q=1}^{|\mathbf{j}|}(-1)^{q-1}T_{j_q}\left[g_{\mathbf{j}}\right]d\bar{z}^{j_1}\wedge\ldots\wedge \\
& \wedge d\bar{z}^{j_{q-1}}\wedge d\bar{z}^{j_{q+1}}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}+ \\
& +\sum_{\mathbf{k}}\sum_{p=1}^{|\mathbf{k}|+1}\sum_{q=k_{p-1}+1}^{k_p-1}(-1)^{p-1}\bar{T}_q\left[g_{\mathbf{k}}\right]d\bar{z}^{k_1}\wedge\ldots\wedge \\
& \wedge d\bar{z}^{k_{p-1}}\wedge d\bar{z}^{q}\wedge d\bar{z}^{k_p}\wedge\ldots\wedge d\bar{z}^{k_{|\mathbf{k}|}}\;+\;H^{11},
\end{aligned}$$

$$\begin{aligned}
F^{21} \;:=\; & \sum_{\mathbf{j}}\sum_{p=1}^{|\mathbf{j}|+1}\sum_{q=j_{p-1}+1}^{j_p-1}(-1)^{p-1}\bar{T}_q\left[g_{\mathbf{j}}\right]d\bar{z}^{j_1}\wedge\ldots\wedge \\
& \wedge d\bar{z}^{j_{p-1}}\wedge d\bar{z}^{q}\wedge d\bar{z}^{j_p}\wedge\ldots\wedge d\bar{z}^{j_{|\mathbf{j}|}}+ \\
& +\sum_{\mathbf{k}}\sum_{q=1}^{|\mathbf{k}|}(-1)^{q-1}\bar{T}_{k_q}\left[g_{\mathbf{k}}\right]d\bar{z}^{k_1}\wedge\ldots\wedge \\
& \wedge d\bar{z}^{k_{q-1}}\wedge d\bar{z}^{k_{q+1}}\wedge\ldots\wedge d\bar{z}^{k_{|\mathbf{k}|}}\;+\;H^{21},
\end{aligned}$$

where H^{11} and H^{21} satisfy the following conditions:

$$\begin{aligned} \bar{\partial}\left[H^{11}\right] + \bar{\partial}^*\left[H^{21}\right] &= 0, \\ \bar{\partial}^*\left[H^{11}\right] + \bar{\partial}\left[H^{21}\right] &= 0. \end{aligned}$$

Thus $F^{11} = F^{22}$ already, and it suffices to take $H^{11} = H^{21} =: h$. Therefore, each solution of the equation (7.5.1) is of the form

$$\begin{aligned} f \;=\; & \sum_{\mathbf{j}} \sum_{p=1}^{|\mathbf{j}|+1} \sum_{q=j_{p-1}+1}^{j_p-1} (-1)^{p-1} \bar{T}_q\left[g_{\mathbf{j}}\right] d\bar{z}^{j_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ & + \sum_{\mathbf{j}} \sum_{q=1}^{|\mathbf{j}|} (-1)^{q-1} \bar{T}_{j_q}\left[g_{\mathbf{j}}\right] d\bar{z}^{j_1} \wedge \ldots \wedge \\ & \wedge d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{j_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} \;+\; h, \end{aligned}$$

where h satisfies the homogeneous Hodge-Dirac system

$$\left(\bar{\partial} + \bar{\partial}^*\right)[h] = 0,$$

that is,

$$\begin{pmatrix} h & 0 \\ h & 0 \end{pmatrix} \in \overline{\mathfrak{N}}\left(\Omega^+\right).$$

Chapter 8

Complex Hodge-Dolbeault system, the $\overline{\partial}$-problem and the Koppelman formula

8.1 Definition of the complex Hodge-Dolbeault system

We consider in this subsection an important particular case of (3.1.1) where $F^{12} = F^{21} = 0$ and $F^{11} = F^{22} =: f$. In this case, (3.1.1) reduces to the system

$$\begin{aligned} \bar{\partial}\,[f] &= 0, \\ \bar{\partial}^*\,[f] &= 0. \end{aligned} \tag{8.1.1}$$

We shall call it the complex Hodge-Dolbeault system by the following reason.

The classical Hodge-de Rham system is the system for harmonic differential forms given by $df = 0$, $d^*f = 0$, "*" being the Hodge duality operator. What is more, the system $\bar{\partial} f = 0$, $\bar{\partial}^* f = 0$ may be seen as a complexification of the previous one.

As moreover the partial system $\bar{\partial} f = 0$ is already called the Dolbeault system and is a complexification of the de Rham system $df = 0$, the name "Hodge-Dolbeault" for the complexification of the "Hodge-de Rham system" seems reasonable.

Let us mention that the examples of hyperholomophic matrices

with neither holomorphic nor antiholomorphic coefficients provide also an example of a solution to the complex Hodge-Dolbeault system which is a differential form, not a function. Indeed, if

$$f := e^{2(Rez_1 - iImz_2)} \left(d\bar{z}^1 + d\bar{z}^2 \right)$$

then $\bar{\partial} f = \bar{\partial}^* f = 0$.

Note that if we are looking for solutions that are functions, not differential forms, then the system (8.1.1) reduces to the equation $\bar{\partial}[f] = 0$, which defines the notion of a holomorphic function in $\mathbb{C}^m$. Hence, at least formally, one may consider the set of all solutions of the system (8.1.1) as a generalization of that notion onto the level of $(0, k)$-differential forms. We shall see in this chapter that such a generalization is not only a formal one but also fits quite consistently certain very well-known objects of complex analysis —first of all, the Koppelman integral representation for smooth differential forms.

Conditions (8.1.1) mean that we are interested in hyperholomorphic **m.v.d.f.** of the form $\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = fE_{2\times 2}$. Hence, to obtain main theorems and to detect, at the same time, peculiarities of the case we shall use the already proved facts. We believe that direct proofs, without appealing to hyperholomorphic theory, would not be so easy.

8.2 Relation with hyperholomorphic case

Let $p \in \{0, \ldots, m\}$ be fixed. Let **d.f.** $\sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ satisfy the complex Hodge-Dolbeault system:

$$\begin{aligned} \bar{\partial} \left[\sum_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} \right] &= 0, \\ \bar{\partial}^* \left[\sum_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} \right] &= 0. \end{aligned}$$

Set $G := \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} E_{2\times 2}$, then G satisfies the formula (4.1.1), hence

$$\int\limits_{\Gamma} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} \sum_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} E_{2\times 2} = 0,$$

that is,

$$\int\limits_{\Gamma} \begin{pmatrix} \bar{\sigma} \wedge \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}; & \bar{\sigma}^* \wedge \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} \\ \bar{\sigma}^* \wedge \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}; & \bar{\sigma} \wedge \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} \end{pmatrix} = 0,$$

which is equivalent to

$$\int\limits_{\Gamma} \bar{\sigma} \wedge \sum_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} \;=\; 0, \tag{8.2.1}$$

$$\int\limits_{\Gamma} \bar{\sigma}^* \wedge \sum_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} \;=\; 0, \tag{8.2.2}$$

where $\bar{\sigma}$ and $\bar{\sigma}^*$ were introduced in Subsection 2.5. Applying also equality (2.5.1) we obtain an equivalent to (8.2.1) and (8.2.2) pair of equalities:

$$\sum_{|\mathbf{j}|=p} \sum_{j=1}^{m} \left(\int\limits_{\Gamma} n_j(\zeta) g_{\mathbf{j}}(\zeta) ds_{\zeta} \right) d\bar{z}^j \wedge d\bar{z}^{\mathbf{j}} \;=\; 0,$$

$$\sum_{|\mathbf{j}|=p} \sum_{j=1}^{m} \left(\int\limits_{\Gamma} \bar{n}_j(\zeta) g_{\mathbf{j}}(\zeta) ds_{\zeta} \right) \widehat{d\bar{z}^j} \wedge d\bar{z}^{\mathbf{j}} \;=\; 0.$$

Now Theorem 4.1 implies the next corollary.

8.3 The Cauchy integral theorem for solutions of degree p for the complex Hodge-Dolbeault system

Corollary *Let $p \in \{0,\, 1,\, \ldots,\, m\}$ be fixed. Let $g = \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ be a p-***d.f.** *of class C^1, satisfying the complex Hodge-Dolbeault system in the*

bounded domain Ω^+ and continuous up to its piecewise smooth boundary. Then in Ω^+ there hold

$$\begin{cases} \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} c_j \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta) d\bar{\zeta}_{[j]} \wedge d\zeta \wedge d\bar{z}^j \wedge d\bar{z}^{\mathbf{j}} = 0, \\ \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} c_j \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta) d\bar{\zeta} \wedge d\bar{\zeta}_{[j]} \wedge \widehat{d\bar{z}^j} \wedge d\bar{z}^{\mathbf{j}} = 0, \end{cases} \tag{8.3.1}$$

or equivalently

$$\begin{cases} \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} \left(\int\limits_{\Gamma} n_j(\zeta) g_j(\zeta) ds_\zeta \right) d\bar{z}^j \wedge d\bar{z}^{\mathbf{j}} = 0, \\ \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} \left(\int\limits_{\Gamma} \bar{n}_j(\zeta) g_j(\zeta) ds_\zeta \right) \widehat{d\bar{z}^j} \wedge d\bar{z}^{\mathbf{j}} = 0. \end{cases} \tag{8.3.2}$$

Recall that here $\mathbf{n}(\zeta) = (n_1(\zeta) \ldots, n_m(\zeta))$ denotes the outward-pointing unit normal vector, and $\bar{\mathbf{n}}(\zeta)$ is its complex conjugate; $\{c_j\}$ are constants defined in Subsection 2.5. Let us mention also that it is possible to formulate (8.3.1) and (8.3.2) more explicitly in terms of coefficients $g_{\mathbf{j}}$. Indeed, with $\mathbf{j} = (j_1, j_2, \ldots, j_p)$ (8.3.1) takes the form

$$\sum_{q=0}^{p-1} (-1)^q \sum_{j_q < \beta < j_{q+1}} c_\beta \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta) d\bar{\zeta}_{[\beta]} \wedge d\zeta \wedge d\bar{z}^{j_1} \wedge \ldots \wedge$$

$$\wedge d\bar{z}^{j_q} \wedge d\bar{z}^{\beta} \wedge d\bar{z}^{j_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_p} = 0,$$

$$\sum_{q=1}^{p} (-1)^{m+q-1} c_{j_q} \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta) d\bar{\zeta} \wedge d\zeta_{[j_q]} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge$$

$$\wedge d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{j_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_p} = 0,$$

and (8.3.2) takes the form

$$\sum_{q=0}^{p-1} (-1)^q \sum_{j_q < \beta < j_{q+1}} \frac{1}{2} \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta) n_\beta(\zeta) dS_\zeta d\bar{z}^{j_1} \wedge \ldots \wedge$$

$$\wedge\, d\bar{z}^{j_q} \wedge d\bar{z}^{\beta} \wedge\ d\bar{z}^{j_{q+1}} \wedge\ \ldots\ \wedge d\bar{z}^{j_p}\ =\ 0,$$

$$\sum_{q=1}^{p}(-1)^{m+q-1}\frac{1}{2}c_{j_q}\int\limits_{\Gamma} g_{\mathbf{j}}(\zeta)\bar{n}_{j_q}(\zeta)dS_{\zeta}d\bar{z}^{j_1}\wedge\ \ldots\ \wedge$$

$$\wedge\, d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{j_{q+1}} \wedge\ \ldots\ \wedge d\bar{z}^{j_p}\ =\ 0.$$

It suffices now to refer to the "degree of differential forms" arguments, thus eliminating the differentials $d\bar{z}^{j}$.

Corollary 8.3 extends, obviously, onto arbitrary solutions of the complex Hodge-Dolbeault system.

8.4 The Cauchy integral theorem for arbitrary solutions of the complex Hodge-Dolbeault system

Corollary *Let* $g = \sum\limits_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ *be a* **d.f.** *of class* C^1*, satisfying the complex Hodge-Dolbeault system in the bounded domain* Ω^+ *and continuous up to its piecewise smooth boundary. Then*

$$\begin{cases} \sum\limits_{\mathbf{j}}\sum\limits_{j=1}^{m} c_j \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta)d\bar{\zeta}_{[j]}\wedge d\zeta\wedge d\bar{z}^{j}\wedge d\bar{z}^{\mathbf{j}}\ =\ 0, \\ \sum\limits_{\mathbf{j}}\sum\limits_{j=1}^{m} c_j \int\limits_{\Gamma} g_{\mathbf{j}}(\zeta)d\bar{\zeta}\wedge d\zeta_{[j]}\wedge \widehat{d\bar{z}^{j}}\wedge d\bar{z}^{\mathbf{j}}\ =\ 0, \end{cases} \tag{8.4.1}$$

or equivalently

$$\begin{cases} \sum\limits_{\mathbf{j}}\sum\limits_{j=1}^{m}\left(\int\limits_{\Gamma} n_j(\zeta)g_{\mathbf{j}}(\zeta)ds_{\zeta}\right)d\bar{z}^{j}\wedge d\bar{z}^{\mathbf{j}}\ =\ 0, \\ \sum\limits_{\mathbf{j}}\sum\limits_{j=1}^{m}\left(\int\limits_{\Gamma} \bar{n}_j(\zeta)g_{\mathbf{j}}(\zeta)ds_{\zeta}\right)\widehat{d\bar{z}^{j}}\wedge d\bar{z}^{\mathbf{j}}\ =\ 0. \end{cases} \tag{8.4.2}$$

8.5 Morera's theorem for solutions of degree p for the complex Hodge-Dolbeault system

Corollary *Let Ω be a domain in $\mathbb{C}^m$ with the topological boundary Γ which is a piecewise smooth surface, let $g = \sum_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ be a p-***d.f.*** of class C^1 in the domain Ω. If for any bounded piecewise smooth surface-without-boundary $\hat{\Gamma}$, with $\hat{\Gamma} = Fr\left(\hat{\Omega}\right)$, $\hat{\Omega} \subset \Omega$, $\hat{\Gamma} \subset \Omega$, one of the following equivalent conditions holds,*

$$\begin{cases} \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} c_j \int\limits_{\hat{\Gamma}} g_{\mathbf{j}}(\zeta) d\bar{\zeta}_{[j]} \wedge d\zeta \wedge d\bar{z}^j \wedge d\bar{z}^{\mathbf{j}} = 0, \\ \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} c_j \int\limits_{\hat{\Gamma}} g_{\mathbf{j}}(\zeta) d\bar{\zeta} \wedge d\zeta_{[j]} \wedge \widehat{d\bar{z}^j} \wedge d\bar{z}^{\mathbf{j}} = 0, \end{cases} \tag{8.5.1}$$

or

$$\begin{cases} \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} \left(\int\limits_{\hat{\Gamma}} n_j(\zeta) g_j(\zeta) ds_\zeta \right) d\bar{z}^j \wedge d\bar{z}^{\mathbf{j}} = 0, \\ \sum\limits_{|\mathbf{j}|=p} \sum\limits_{j=1}^{m} \left(\int\limits_{\hat{\Gamma}} \bar{n}_j(\zeta) g_j(\zeta) ds_\zeta \right) \widehat{d\bar{z}^j} \wedge d\bar{z}^{\mathbf{j}} = 0, \end{cases} \tag{8.5.2}$$

then g satisfies the complex Hodge-Dolbeault system (8.1.1).

Proof. Using the computation in Subsection 8.2 and combining it with Theorem 5.1, we have that $G := gE_{2\times 2}$ is in $\overline{\mathfrak{N}}(\Omega)$, that is,

$$\overline{\mathcal{D}}[G] = \begin{pmatrix} \bar{\partial}\left[\sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] & \bar{\partial}^*\left[\sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] \\ \bar{\partial}^*\left[\sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] & \bar{\partial}\left[\sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] \end{pmatrix} = 0,$$

equivalently,

$$\bar{\partial}[g] = 0,$$

$$\bar{\partial}^{*}\left[g\right] \;=\; 0,$$

i.e., g satisfies the complex Hodge-Dolbeault system. ■

Note that this Corollary is the inverse theorem for Corollary 8.3. Hence, it establishes an equivalent condition for a given p-**d.f.** $g = \sum\limits_{|\mathbf{j}|=p} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ to satisfy the complex Hodge-Dolbeault system.

8.6 Morera's theorem for arbitrary solutions of the complex Hodge-Dolbeault system

Corollary *Let Ω be a domain in $\mathbb{C}^m$ with the topological boundary Γ which is a piecewise smooth surface, let $g = \sum\limits_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ be a* **d.f.** *of class C^1 in the domain Ω. If for any bounded piecewise smooth surface-without-boundary $\hat{\Gamma}$, with $\hat{\Gamma} = Fr\left(\hat{\Omega}\right)$, $\hat{\Omega} \subset \Omega$, $\hat{\Gamma} \subset \Omega$, one of the following equivalent conditions holds*

$$\left\{\begin{array}{l} \sum\limits_{\mathbf{j}} \sum\limits_{j=1}^{m} c_j \int\limits_{\hat{\Gamma}} g_{\mathbf{j}}(\zeta) d\zeta_{[j]} \wedge d\zeta \wedge d\bar{z}^{j} \wedge d\bar{z}^{\mathbf{j}} \;=\; 0, \\ \\ \sum\limits_{\mathbf{j}} \sum\limits_{j=1}^{m} c_j \int\limits_{\hat{\Gamma}} g_{\mathbf{j}}(\zeta) d\bar{\zeta} \wedge d\zeta_{[j]} \wedge \widehat{d\bar{z}^{j}} \wedge d\bar{z}^{\mathbf{j}} \;=\; 0, \end{array}\right. \tag{8.6.1}$$

or

$$\left\{\begin{array}{l} \sum\limits_{\mathbf{j}} \sum\limits_{j=1}^{m} \left(\int\limits_{\hat{\Gamma}} n_j(\zeta) g_{\mathbf{j}}(\zeta) ds_{\zeta} \right) d\bar{z}^{j} \wedge d\bar{z}^{\mathbf{j}} \;=\; 0, \\ \\ \sum\limits_{\mathbf{j}} \sum\limits_{j=1}^{m} \left(\int\limits_{\hat{\Gamma}} \bar{n}_j(\zeta) g_{\mathbf{j}}(\zeta) ds_{\zeta} \right) \widehat{d\bar{z}^{j}} \wedge d\bar{z}^{\mathbf{j}} \;=\; 0, \end{array}\right. \tag{8.6.2}$$

then g satisfies the complex Hodge-Dolbeault system.

Proof. Mimics the above ones. ■

Note that this Corollary is the inverse theorem for the Corollary 8.4. Hence, it establishes an equivalent condition for a given **d.f.** $g = \sum\limits_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ to satisfy the Hodge-Dolbeault system.

8.7 Solutions of a fixed degree

Let $s = 1, \ldots, m-1$, be fixed. Let $G = \sum_{|\mathbf{j}|=s} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} E_{2\times 2}$ where the **d.f.** $\sum_{|\mathbf{j}|=s} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ satisfies the complex Hodge-Dolbeault system

$$\begin{aligned} \bar{\partial}\left[\sum_{|\mathbf{j}|=s} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] &= 0, \\ \bar{\partial}^*\left[\sum_{|\mathbf{j}|=s} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] &= 0. \end{aligned}$$

Then G satisfies the formula (6.1.1); hence in Ω^+ ,

$$G(z) = \frac{1}{2}\int_{\Gamma} \mathcal{K}_{\bar{\mathcal{D}}}(\tau - z, z) \underset{\star}{\wedge} \bar{\boldsymbol{\sigma}}_{\tau,z} \wedge G(\tau). \tag{8.7.1}$$

Using (2.3.1) we get for the right-hand side

$$\begin{aligned} \sum_{|\mathbf{j}|=s} \Big(& U_{\mathbf{j}}[g_{\mathbf{j}}](z) \cdot d\bar{z}^{\mathbf{j}} \cdot E_{2\times 2} + \\ & + \sum_{q\neq p} \big(U_{qp}[G_{\mathbf{j}}] \widehat{d\bar{z}^q} \wedge d\bar{z}^p \wedge d\bar{z}^{\mathbf{j}} E_{2\times 2} + \\ & \quad + \overline{U}_{qp}[g_{\mathbf{j}}] d\bar{z}^q \wedge \widehat{d\bar{z}^p} \wedge d\bar{z}^{\mathbf{j}} E_{2\times 2} \big) + \\ & + \sum_{q<p} \big(\overline{W}_{qp}[g_{\mathbf{j}}] - \overline{W}_{pq}[g_{\mathbf{j}}] \big) d\bar{z}^q \wedge d\bar{z}^p \wedge d\bar{z}^{\mathbf{j}} \check{E}_{2\times 2} + \\ & + \sum_{q<p} (W_{qp}[g_{\mathbf{j}}] - W_{pq}[g_{\mathbf{j}}]) \widehat{d\bar{z}^q} \wedge \widehat{d\bar{z}^p} \wedge d\bar{z}^{\mathbf{j}} \check{E}_{2\times 2} \Big), \end{aligned} \tag{8.7.2}$$

where the definition of $U_{\mathbf{j}}$ is (2.10.2) and the definitions of U_{qp}, $\overline{U}_{qp}$, W_{qp}, and $\overline{W}_{qp}$ are in Subsection 2.11.8.

8.8 Arbitrary solutions

It is easy to extend the above onto arbitrary solutions of the complex Hodge-Dolbeault system.

Let **d.f.** $\sum\limits_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ satisfy the complex Hodge-Dolbeault system

$$\begin{aligned} \bar{\partial}\left[\sum_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] &= 0, \\ \bar{\partial}^{*}\left[\sum_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}\right] &= 0, \end{aligned}$$

and let again $G := \sum\limits_{\mathbf{j}} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}} E_{2\times 2}$. Then (6.1.1) holds in Ω^{+}, and for its right-hand-side we obtain

$$\begin{aligned} \sum_{\mathbf{j}} \quad & \Big(U_{\mathbf{j}}[g_{\mathbf{j}}](z) \cdot d\bar{z}^{\mathbf{j}} \cdot E_{2\times 2} \; + \\ & + \sum_{q\neq p} \big(U_{qp}[g_{\mathbf{j}}] \widehat{d\bar{z}^{q}} \wedge d\bar{z}^{p} \wedge d\bar{z}^{\mathbf{j}} E_{2\times 2} + \\ & \quad + \overline{U}_{qp}[g_{\mathbf{j}}] d\bar{z}^{q} \wedge \widehat{d\bar{z}^{p}} \wedge d\bar{z}^{\mathbf{j}} E_{2\times 2} \big) \; + \\ & + \sum_{q<p} \big(\overline{W}_{qp}[g_{\mathbf{j}}] \; - \; \overline{W}_{pq}[g_{\mathbf{j}}] \big) d\bar{z}^{q} \wedge d\bar{z}^{p} \wedge d\bar{z}^{\mathbf{j}} \check{E}_{2\times 2} \; + \\ & + \sum_{q<p} (W_{qp}[g_{\mathbf{j}}] \; - \; W_{pq}[g_{\mathbf{j}}]) \widehat{d\bar{z}^{q}} \wedge \widehat{d\bar{z}^{p}} \wedge d\bar{z}^{\mathbf{j}} \check{E}_{2\times 2} \Big). \end{aligned} \tag{8.8.1}$$

Taking into account the structure of the matrices $E_{2\times 2}$ and $\check{E}_{2\times 2}$ we arrive at the following corollary.

8.9 Bochner-Martinelli-type integral representation for solutions of degree s of the complex Hodge-Dolbeault system

Corollary *Let $s = 1, \ldots, m$, be fixed. Let $g = \sum\limits_{|\mathbf{j}|=s} g_{\mathbf{j}} d\bar{z}^{\mathbf{j}}$ be an s-*

d.f. *of class C^{1}, satisfying the complex Hodge-Dolbeault system in the*

bounded domain Ω^+, which can be extended continuously up to the topological piecewise smooth boundary Γ of Ω^+. Then

$$g(z) = \sum_{|\mathbf{j}|=s}\Bigg(U_{\mathbf{j}}[g_{\mathbf{j}}](z)\cdot d\bar{z}^{\mathbf{j}} + \sum_{q\neq p}\big(U_{qp}[g_{\mathbf{j}}]\,\widehat{d\bar{z}^q}\wedge d\bar{z}^p\wedge d\bar{z}^{\mathbf{j}} + \overline{U}_{qp}[g_{\mathbf{j}}]\,d\bar{z}^q\wedge\widehat{d\bar{z}^p}\wedge d\bar{z}^{\mathbf{j}}\big)\Bigg); \tag{8.9.1}$$

$$\sum_{|\mathbf{j}|=s}\Bigg(\sum_{q<p}\left(\overline{W}_{qp}[g_{\mathbf{j}}] - \overline{W}_{pq}[g_{\mathbf{j}}]\right)d\bar{z}^q\wedge d\bar{z}^p\wedge d\bar{z}^{\mathbf{j}} + \sum_{q<p}\left(W_{qp}[g_{\mathbf{j}}] - W_{pq}[g_{\mathbf{j}}]\right)\widehat{d\bar{z}^q}\wedge\widehat{d\bar{z}^p}\wedge d\bar{z}^{\mathbf{j}}\Bigg) = 0. \tag{8.9.2}$$

Although formula (8.9.1) has been derived from the Cauchy integral representation, we prefer to use the name "the **Bochner-Martinelli** integral representation", since the theory of null-solutions of the complex Hodge-Dolbeault system differs essentially from the one-dimensional complex function theory and has closer analogy with the structure of several complex variables theory. Let us mention also that the equality (8.9.2) is not true, generally speaking, for a given differential form $\sum_{|\mathbf{j}|=s} g_{\mathbf{j}}dz^{\mathbf{j}}$ of class C^1; what is more, the conditions (8.9.1) and (8.9.2) are characteristic for solutions of the complex Hodge-Dolbeault system.

8.10 Bochner-Martinelli-type integral representation for arbitrary solutions of the complex Hodge-Dolbeault system

Corollary *Let $g = \sum_{\mathbf{j}} g_{\mathbf{j}}d\bar{z}^{\mathbf{j}}$ be a* **d.f.** *of class C^1, satisfying the complex Hodge-Dolbeault system in the bounded domain Ω^+, which can be extended continuously up to the topological, piecewise smooth boundary Γ of Ω^+. Then in Ω^+,*

$$g(z) = \sum_{\mathbf{j}}\big(U_{\mathbf{j}}[g_{\mathbf{j}}](z)\cdot d\bar{z}^{\mathbf{j}} +$$

$$+\sum_{q\neq p}\Big(U_{qp}\left[g_{\mathbf{j}}\right]\widehat{d\bar{z}^{q}}\wedge d\bar{z}^{p}\wedge d\bar{z}^{\mathbf{j}}\ +\ \overline{U}_{qp}\left[g_{\mathbf{j}}\right]d\bar{z}^{q}\wedge\widehat{d\bar{z}^{p}}\wedge d\bar{z}^{\mathbf{j}}\Big)\,;\tag{8.10.1}$$

$$\sum_{q<p}\left(\overline{W}_{qp}\left[g_{\mathbf{j}}\right]\ -\ \overline{W}_{pq}\left[g_{\mathbf{j}}\right]\right)d\bar{z}^{q}\wedge d\bar{z}^{p}\wedge d\bar{z}^{\mathbf{j}}\ +$$

$$+\sum_{q<p}\left(W_{qp}\left[g_{\mathbf{j}}\right]\ -\ W_{pq}\left[g_{\mathbf{j}}\right]\right)\widehat{d\bar{z}^{q}}\wedge\widehat{d\bar{z}^{p}}\wedge d\bar{z}^{\mathbf{j}}\quad=\quad 0.\tag{8.10.2}$$

The analogous comments can be repeated.

8.11 Solution of the $\bar{\partial}$-type problem for the complex Hodge-Dolbeault system in a bounded domain in $\mathbb{C}^m$

Let Ω^{+} and Γ be as always. Consider the $\bar{\partial}$-type problem for the complex Hodge-Dolbeault system, that is,

$$\begin{array}{rl}\bar{\partial}\left[f\right]= & g^{1},\\ \bar{\partial}^{*}\left[f\right]= & g^{2},\end{array}\tag{8.11.1}$$

where $g^{1},\ g^{2}\in\overline{\mathcal{G}}_{1}\left(\Omega^{+}\right)\cap\overline{\mathcal{G}}_{0}\left(\Omega^{+}\cup\Gamma\right)$, $g^{1}=\sum\limits_{\mathbf{j}}g^{1}_{\mathbf{j}}d\bar{z}^{\mathbf{j}};\ g^{2}=\sum\limits_{\mathbf{k}}g^{2}_{\mathbf{k}}d\bar{z}^{\mathbf{k}}$.

Comparing (8.11.1) and (7.4.1), one can conclude that solutions of (8.11.1) are exactly all solutions $(F^{11}=f;\ F^{21})$ of (7.4.1), with $G^{11}:=g^{1}$ and $G^{21}:=g^{2}$, such that $\overline{\partial}^{*}\left[F^{21}\right]\equiv\overline{\partial}\left[F^{21}\right]\equiv 0$. Recalling formulas (7.4.4) we arrive at a **necessary and sufficient condition** for the problem (8.11.1) to be solvable: the system (8.11.1) has a solution if, and only if, the differential form

$$\sum_{\mathbf{j}}\sum_{p=1}^{|\mathbf{j}|+1}\sum_{q=j_{p-1}+1}^{j_{p}-1}(-1)^{p-1}\overline{T}_{q}\left[g^{1}_{\mathbf{j}}\right]d\bar{z}^{j_{1}}\wedge\ \ldots\ \wedge$$

$$\wedge\,d\bar{z}^{j_{p-1}}\wedge d\bar{z}^{q}\wedge d\bar{z}^{j_{p}}\wedge\ \ldots\ \wedge d\bar{z}^{j_{|\mathbf{j}|}}+$$

$$+ \sum_{\mathbf{k}} \sum_{q=1}^{|\mathbf{k}|} (-1)^{q-1} T_{k_q} \left[g_{\mathbf{j}}^2 \right] d\bar{z}^{k_1} \wedge \ldots \wedge$$
$$\wedge d\bar{z}^{k_{q-1}} \wedge d\bar{z}^{k_{q+1}} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} \tag{8.11.2}$$

satisfies the complex Hodge-Dolbeault system (see (8.1.1)). Given g^1, g^2 with the property (8.11.2), appealing again to (7.4.4) we obtain that each solution of the system (8.11.1) is of the form

$$f = \sum_{\mathbf{j}} \sum_{q=1}^{|\mathbf{j}|} (-1)^{q-1} T_{j_q} \left[g_{\mathbf{j}}^1 \right] d\bar{z}^{j_1} \wedge \ldots \wedge$$
$$\wedge d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{j_{q+1}} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} +$$

$$+ \sum_{\mathbf{k}} \sum_{p=1}^{|\mathbf{k}|+1} \sum_{q=k_{p-1}+1}^{k_p - 1} (-1)^{p-1} \overline{T}_q \left[g_{\mathbf{j}}^2 \right] d\bar{z}^{k_1} \wedge \ldots \wedge$$
$$\wedge d\bar{z}^{k_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{k_p} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} + h \tag{8.11.3}$$

where h is an arbitrary null-solution of the homogeneous Hodge-Dolbeault system.

We see again that there exists a deep difference between hyperholomorphic theory and the theory of the complex Hodge-Dolbeault system: while the hypercomplex $\overline{\mathcal{D}}$-problem is always solvable, its analog (and a particular case) (8.11.1) has a necessary and sufficient condition for its solvability. For this reason, we have chosen the term "$\overline{\partial}$-type problem" instead of something like "$\overline{\partial} - \overline{\partial}^*$-problem."

8.12 Complex $\bar{\partial}$-problem and the $\bar{\partial}$-type problem for the complex Hodge-Dolbeault system

Consider the $\bar{\partial}$-problem, that is, to find all **d.f.** which satisfy the equation

$$\bar{\partial}[f] = g, \tag{8.12.1}$$

where $g \in \overline{\mathcal{G}}_1(\Omega^+) \cap \overline{\mathcal{G}}_0(\Omega^+ \cup \Gamma)$.

Let g be such that (8.12.1) has solutions, and let f_0 be one of them. Consider the $\bar{\partial}$-type problem

$$\begin{aligned} \bar{\partial}\,[f] &= g, \\ \bar{\partial}^*\,[f] &= \bar{\partial}^*\,[f_0]\,, \end{aligned} \tag{8.12.2}$$

with f an unknown **d.f.** Since (8.12.2) has an obvious solution, f_0, then it follows from Subsection 8.11 that the **d.f.**

$$\begin{aligned} &\sum_{\mathbf{j}} \sum_{p=1}^{|\mathbf{j}|+1} \sum_{q=j_{p-1}+1}^{j_p-1} (-1)^{p-1}\, \overline{T}_q\,[g_{\mathbf{j}}]\, d\bar{z}^{j_1} \wedge \ldots \wedge \\ &\qquad \wedge\, d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ &+ \sum_{\mathbf{k}} \sum_{q=1}^{|\mathbf{k}|} (-1)^{q-1}\, T_{k_q}\left[\bar{\partial}^*\,[f_0]_{\mathbf{k}}\right] d\bar{z}^{k_1} \wedge \ldots \wedge \\ &\qquad \wedge\, d\bar{z}^{k_{q-1}} \wedge d\bar{z}^{k_{q+1}} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} \end{aligned} \tag{8.12.3}$$

satisfies the complex Hodge-Dolbeault system.

This can be seen as follows. If g is such that (8.12.1) has a solution, then there exists a **d.f.** $\tilde{g}$ with the following property: the **d.f.**

$$\begin{aligned} &\sum_{\mathbf{j}} \sum_{p=1}^{|\mathbf{j}|+1} \sum_{q=j_{p-1}+1}^{j_p-1} (-1)^{p-1}\, \overline{T}_q\,[g_{\mathbf{j}}]\, d\bar{z}^{j_1} \wedge \ldots \wedge \\ &\qquad \wedge\, d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\ &+ \sum_{\mathbf{k}} \sum_{q=1}^{|\mathbf{k}|} (-1)^{q-1}\, T_{k_q}\left[\tilde{g}_{\mathbf{k}}\right] d\bar{z}^{k_1} \wedge \ldots \wedge \\ &\qquad \wedge\, d\bar{z}^{k_{q-1}} \wedge d\bar{z}^{k_{q+1}} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} \end{aligned} \tag{8.12.4}$$

satisfies the complex Hodge-Dolbeault system.

Let us prove now that the above condition is sufficient also, i.e., if there exists a **d.f.** $\tilde{g}$ which satisfies (8.12.4) then (8.12.1) has a solution. Really, if there exists such $\tilde{g}$, the $\bar{\partial}$-type problem

$$\begin{cases} \bar{\partial}\,[f] &= g, \\ \bar{\partial}^*\,[f] &= \tilde{g}, \end{cases}$$

has a solution f_0 which is immediately a solution to (8.12.1).

We resume all the above reasonings in the following theorems.

8.13 $\bar{\partial}$-problem for differential forms

Theorem *Let Ω be a domain with the piecewise smooth boundary Γ, and let $g \in \overline{\mathcal{G}}_1(\Omega^+) \cap \overline{\mathcal{G}}_0(\Omega^+ \cup \Gamma)$. The equation*

$$\bar{\partial}[f] = g$$

has a solution if and only if there exists a **d.f.** $\tilde{g} = \sum_{\mathbf{k}} \tilde{g}_{\mathbf{k}} d\bar{z}^{\mathbf{k}}$ *such that the* **d.f.**

$$\begin{aligned}
&\sum_{\mathbf{j}} \sum_{p=1}^{|\mathbf{j}|+1} \sum_{q=j_{p-1}+1}^{j_p-1} (-1)^{p-1} \overline{T}_q [g_{\mathbf{j}}] \, d\bar{z}^{j_1} \wedge \ldots \wedge \\
&\qquad \wedge d\bar{z}^{j_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\
&+ \sum_{\mathbf{k}} \sum_{q=1}^{|\mathbf{k}|} (-1)^{q-1} T_{k_q} [\tilde{g}_{\mathbf{k}}] \, d\bar{z}^{k_1} \wedge \ldots \wedge \\
&\qquad \wedge d\bar{z}^{k_{q-1}} \wedge d\bar{z}^{k_{q+1}} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}}
\end{aligned} \tag{8.13.1}$$

satisfies the complex Hodge-Dolbeault system. If it is true, then each solution f of the equation (8.12.1) is of the form

$$\begin{aligned}
f \;=\; &\sum_{\mathbf{j}} \sum_{q=1}^{|\mathbf{j}|} (-1)^{q-1} T_{j_q} [g_{\mathbf{j}}] \, d\bar{z}^{j_1} \wedge \ldots \wedge \\
&\qquad \wedge d\bar{z}^{j_{q-1}} \wedge d\bar{z}^{q+1} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} + \\
+ \;&\sum_{\mathbf{k}} \sum_{p=1}^{|\mathbf{k}|+1} \sum_{q=\mathbf{j}_{p-1}+1}^{\mathbf{j}_p-1} (-1)^{p-1} \overline{T}_q [\tilde{g}_{\mathbf{k}}] \, d\bar{z}^{k_1} \wedge \ldots \wedge \\
&\qquad \wedge d\bar{z}^{k_{p-1}} \wedge d\bar{z}^{q} \wedge d\bar{z}^{k_p} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}} + h,
\end{aligned} \tag{8.13.2}$$

where h is an arbitrary null-solution of the operator $\bar{\partial}$, that is

$$\bar{\partial}[h] \equiv 0.$$

8.13.1 $\bar{\partial}$-problem for functions of several complex variables

Theorem *Let Ω^+ be a bounded open set in $\mathbb{C}^m$ with the topological boundary Γ, which is a piecewise smooth surface. Let $g \in \overline{\mathcal{G}}_2^1(\Omega^+) \cap \overline{\mathcal{G}}_1^0(\Omega^+ \cup \Gamma)$. Then the $\bar{\partial}$-problem*

$$\bar{\partial}[f] = g, \tag{8.13.3}$$

has a solution in $\overline{\mathcal{G}}_1^0(\Omega^+)$ if, and only if, the **d.f.**

$$\sum_{q,j} \overline{T}_q[g_j]\, d\bar{z}^q \wedge d\bar{z}^j \tag{8.13.4}$$

satisfies the complex Hodge-Dolbeault system (see (8.1.1)). If it holds then each solution $f \in \overline{\mathcal{G}}_1^0(\Omega^+)$ of the equation (8.13.3) is of the form

$$f = \sum_{q=1}^{m} T_q[g_q] + h, \tag{8.13.5}$$

where h is an arbitrary holomorphic function in Ω.

Proof. If $f_0 \in \overline{\mathcal{G}}_1^0(\Omega^+)$ is a solution of the equation (8.13.3), then necessarily $\bar{\partial}^*[f_0] \equiv 0$. Hence $\tilde{g} \equiv 0$ can be taken in Theorem 8.13, and the condition (8.13.1) converts into

$$\sum_{j=1}^{m} \sum_{p=1}^{2} \sum_{q=j_{p-1}+1}^{j_p - 1} (-1)^{p-1}\, \overline{T}_q[g_j]\, d\bar{z}^{j_1} \wedge \ldots \wedge$$

$$\wedge\, d\bar{z}^{j_{p-1}} \wedge d\bar{z}^q \wedge d\bar{z}^{j_p} \wedge \ldots \wedge d\bar{z}^{j_{|\mathbf{j}|}} =$$

$$= \sum_{j=1}^{m} \left(\sum_{q=1}^{j-1} \overline{T}_q[g_j]\, d\bar{z}^q \wedge d\bar{z}^j - \sum_{q=j+1}^{m} \overline{T}_q[g_j]\, d\bar{z}^j \wedge d\bar{z}^q \right) =$$

$$= \sum_{j=1}^{m} \left(\sum_{q=1}^{j-1} \overline{T}_q[g_j]\, d\bar{z}^q + \sum_{q=j+1}^{m} \overline{T}_q[g_j]\, d\bar{z}^q \right) \wedge d\bar{z}^j =$$

$$= \sum_{j=1}^{m} \sum_{q=1}^{m} \overline{T}_q[g_j]\, d\bar{z}^q \wedge d\bar{z}^j,$$

which gives (8.13.4). Analogously, (8.13.2) is transformed into (8.13.5). ■

8.14 General situation of the Borel-Pompeiu representation

In Subsections 8.7 to 8.10 we made use of the hyperholomorphic Cauchy integral representation (6.1.1), which brought us to the integral representations (8.9.1) and (8.10.1) where only the surface integrals are included, not the volume ones. Consider now a more general situation of the Borel-Pompeiu integral representation (2.11.1), which was written in Subsection 7.1 as

$$2F(z) \;=\; \mathcal{K}_{\overline{\mathcal{D}}}[F](z) \;+\; T_{\overline{\mathcal{D}}} \circ \overline{\mathcal{D}}[F](z),$$

with

$$\mathcal{K}_{\overline{\mathcal{D}}}[F](z) \;:=\; \int\limits_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}}\left(\tau - z,\; z\right) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F\left(\tau,\; d\bar{z}\right),$$

$$T_{\overline{\mathcal{D}}}[F](z) \;:=\; -\int\limits_{\Omega^{+}} \mathcal{K}_{\overline{\mathcal{D}}}\left(\tau - z,\; z\right) \underset{\star}{\wedge} F\left(\tau,\; d\bar{z}\right) dV_{\tau}.$$

The following formula was proved in Subsection 2.10:

$$\begin{aligned}
&\mathcal{K}_{\overline{\mathcal{D}}}\left(\tau - z,\; z\right) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} = \\
&= \;\; (-1)^{\frac{m(m-1)}{2}}\, 2 \left\{ \sum_{j=1}^{m} (-1)^{j-1}\, \mathcal{U}_{j}\left(\tau,\; z\right) d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^{j}} \wedge \right. \\
&\qquad\qquad \left. \wedge\, d\bar{z}^{j}\, E_{2\times 2} \;+ \right. \\
&\quad + (-1)^{m} \sum_{j=1}^{m} (-1)^{j-1}\, \overline{\mathcal{U}}_{j}\left(\tau,\; z\right) d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^{j} \wedge \widehat{d\bar{z}^{j}}\, E_{2\times 2} \;+ \\
&\quad + \sum_{q<j} \Big((-1)^{j-1}\, \overline{\mathcal{U}}_{q}\left(\tau,\; z\right) d\bar{\tau}_{[j]} \wedge d\tau \;-
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{q-1} \overline{\mathcal{U}}_j\,(\tau,\ z)\, d\bar{\tau}_{[q]} \wedge d\tau \Big) \wedge d\bar{z}^q \wedge d\bar{z}^j \breve{E}_{2\times 2} + \\
&+ (-1)^m \sum_{q<j} \Big((-1)^{j-1} \mathcal{U}_q\,(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} - \\
&- (-1)^{q-1} \mathcal{U}_j\,(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[q]} \Big) \wedge \widehat{d\bar{z}^q} \wedge \widehat{d\bar{z}^j} \breve{E}_{2\times 2} + \\
&+ \sum_{q\neq j} (-1)^{j-1} \mathcal{U}_q\,(\tau,\ z)\, d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^q} \wedge d\bar{z}^j E_{2\times 2} + \\
&+ (-1)^m \sum_{j\neq q} (-1)^{j-1} \overline{\mathcal{U}}_q\,(\tau,\ z)\, d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^q \wedge \widehat{d\bar{z}^j} E_{2\times 2} \Bigg\},
\end{aligned}
$$

which means that the product $\mathcal{K}_{\overline{\mathcal{D}}}(\tau\ -\ z,\ z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\,z}$ is the matrix

$$
\begin{pmatrix} \mathcal{A}_1; & \mathcal{A}_2 \\ \mathcal{A}_2; & \mathcal{A}_1 \end{pmatrix} :=\ \mathcal{A}(\tau,\ z)
$$

where

$$
\begin{aligned}
\mathcal{A}_1 \ :=\ & (-1)^{\frac{m(m-1)}{2}} 2 \Big\{ \sum_{j=1}^{m} (-1)^{j-1} \mathcal{U}_j(\tau, z) d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^j} \wedge d\bar{z}^j + \\
& + (-1)^m \sum_{j=1}^{m} (-1)^{j-1} \overline{\mathcal{U}}_j(\tau, z) d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^j \wedge \widehat{d\bar{z}^j} + \\
& + \sum_{q\neq j} (-1)^{j-1} \mathcal{U}_q(\tau, z) d\bar{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\bar{z}^q} \wedge d\bar{z}^j + \\
& + (-1)^m \sum_{j\neq q} (-1)^{j-1} \overline{\mathcal{U}}_q(\tau, z) d\bar{\tau} \wedge d\tau_{[j]} \wedge d\bar{z}^q \wedge \widehat{d\bar{z}^j} \Big\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_2 \ :=\ & (-1)^{\frac{m(m-1)}{2}} 2 \Big\{ \sum_{q<j} \big((-1)^{j-1} \overline{\mathcal{U}}_q(\tau, z) d\bar{\tau}_{[j]} \wedge d\tau - \\
& - (-1)^{q-1} \overline{\mathcal{U}}_j(\tau, z) d\bar{\tau}_{[q]} \wedge d\tau \big) \wedge d\bar{z}^q \wedge d\bar{z}^j \ +
\end{aligned}
$$

$$+ (-1)^m \sum_{q<j} \big((-1)^{j-1} \mathcal{U}_q(\tau,\ z) d\bar{\tau} \wedge d\tau_{[j]} \ -$$

$$- (-1)^{q-1} \mathcal{U}_j(\tau, z) d\bar{\tau} \wedge d\tau_{[q]} \big) \wedge \widehat{d\bar{z}^q} \wedge \widehat{d\bar{z}^j} \Big\}.$$

Hence, the Cauchy-type integral of a **m.v.d.f.** F is an integral operator over Γ with the kernel $\mathcal{A}(\tau,\ z)$ (being itself an operator!):

$$K_{\overline{\mathcal{D}}}[F](z) \ = \int_{\Gamma} \mathcal{A}(\tau,\ z) \ \wedge \ F(\tau,\ dz).$$

For the second term in the Borel-Pompeiu formula we have from Subsection 2.11.1

$$\mathcal{K}_{\overline{\mathcal{D}}}(\tau - z,\ z) \underset{\star}{\wedge} \overline{\mathcal{D}}\,[F]\,(\tau,\ d\bar{z}) =: R = \left(R^{ij}\right),$$

where

$$R_{11} \ = \ 2\frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q} \wedge \big(\bar{\partial} \left[F^{11}\right](\tau,\ d\bar{z}) +$$

$$+ \bar{\partial}^* \left[F^{21}\right](\tau,\ d\bar{z}) \big) +$$

$$+ 2\frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q \wedge \big(\bar{\partial}^* \left[F^{11}\right](\tau,\ d\bar{z}) +$$

$$+ \bar{\partial} \left[F^{21}\right](\tau,\ d\bar{z}) \big),$$

$$R_{21} \ = \ 2\frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \frac{\tau_q - z_q}{|\tau - z|^{2m}} d\bar{z}^q \wedge \big(\bar{\partial} \left[F^{11}\right](\tau,\ d\bar{z}) +$$

$$+ \bar{\partial}^* \left[F^{21}\right](\tau, d\bar{z}) \big) +$$

$$+ 2\frac{(m-1)!}{\pi^m} \sum_{q=1}^{m} \frac{\bar{\tau}_q - \bar{z}_q}{|\tau - z|^{2m}} \widehat{d\bar{z}^q} \wedge \big(\bar{\partial}^* \left[F^{11}\right](\tau,\ d\bar{z}) +$$

$$+\bar{\partial}\left[F^{21}\right](\tau,\ d\bar{z})\big)\,,$$

$$\begin{aligned}
R_{12} &= 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q-\bar{z}_q}{|\tau-z|^{2m}}\widehat{d\bar{z}^q}\wedge\big(\bar{\partial}\left[F^{12}\right](\tau,\ d\bar{z})+\\
&\quad+\bar{\partial}^*\left[F^{22}\right](\tau,\ d\bar{z})\big)+\\
&\quad+2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar{z}^q\wedge\big(\bar{\partial}^*\left[F^{12}\right](\tau,\ d\bar{z})+\\
&\quad+\bar{\partial}\left[F^{22}\right](\tau,d\bar{z})\big)\,,
\end{aligned}$$

$$\begin{aligned}
R_{22} &= 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar{z}^q\wedge\big(\bar{\partial}\left[F^{12}\right](\tau,\ d\bar{z})+\\
&\quad+\bar{\partial}^*\left[F^{22}\right](\tau,\ d\bar{z})\big)\\
&\quad+2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q-\bar{z}_q}{|\tau-z|^{2m}}\widehat{d\bar{z}^q}\wedge\big(\bar{\partial}^*\left[F^{12}\right](\tau,\ d\bar{z})+\\
&\quad+\bar{\partial}\left[F^{22}\right](\tau,\ d\bar{z})\big)\,.
\end{aligned}$$

Hence the Borel-Pompeiu formula takes the form

$$2F(z)\ =\ \int_{\Gamma}\mathcal{A}(\tau,\ z)\ \wedge\ F(\tau,\ d\bar{z})\ -\ \int_{\Omega^+}R(\tau,\ z;d\bar{z})dV_\tau.\qquad(8.14.1)$$

Let f be a **d.f.**, and set $F\ :=\ fE_{2\times 2}$. Then

$$\begin{aligned}
R_{11} &= 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar{\tau}_q-\bar{z}_q}{|\tau-z|^{2m}}\widehat{d\bar{z}^q}\wedge\bar{\partial}\left[f\right](\tau,\ d\bar{z})+\\
&\quad+2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar{z}^q\wedge\bar{\partial}^*\left[f\right](\tau,\ d\bar{z})\,;
\end{aligned}$$

$$
\begin{aligned}
R_{21} \;=\;& 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge\bar\partial\,[f]\,(\tau,\,d\bar z)+\\
&+2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge\bar\partial^*\,[f]\,(\tau,\,d\bar z)\,;\\
R_{12} \;=\;& 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge\bar\partial^*\,[f]\,(\tau,\,d\bar z)\;+\\
&+2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge\bar\partial\,[f]\,(\tau,\,d\bar z)\,;\\
R_{22} \;=\;& 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\tau_q-z_q}{|\tau-z|^{2m}}d\bar z^q\wedge\bar\partial^*\,[f]\,(\tau,\,d\bar z)\;+\\
&+2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\frac{\bar\tau_q-\bar z_q}{|\tau-z|^{2m}}\widehat{d\bar z^q}\wedge\bar\partial\,[f]\,(\tau,\,d\bar z)\,,
\end{aligned}
\tag{8.14.2}
$$

that is, in particular, $R_{11}=R_{22}$, $R_{21}=R_{12}$, and the formula (8.14.1) written component-wisely gives the following equalities:

$$
\begin{aligned}
2f \;&=\; \int\limits_{\Gamma}\mathcal{A}_1\wedge f\;-\int\limits_{\Omega^+}R_{11}(\tau,\,z;d\bar z)dV_\tau;\\
0 \;&=\; \int\limits_{\Gamma}\mathcal{A}_2\wedge f\;-\int\limits_{\Omega^+}R_{12}(\tau,\,z;d\bar z)dV_\tau,
\end{aligned}
$$

or, more explicitly,

$$
\begin{aligned}
&f(z,d\bar z)=(-1)^{\frac{m(m-1)}{2}}\cdot\\
&\quad\cdot\int\limits_{\Gamma}\Bigg\{\sum_{j=1}^{m}(-1)^{j-1}\mathcal{U}_j(\tau,z)d\bar\tau_{[j]}\wedge d\tau\wedge\widehat{d\bar z^j}\wedge d\bar z^j\wedge f(\tau,d\bar z)+
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^m \sum_{j=1}^{m}(-1)^{j-1}\overline{\mathcal{U}}_j(\tau,z)d\bar{\tau}\wedge d\tau_{[j]}\wedge d\bar{z}^j\wedge\widehat{d\bar{z}^j}\wedge\\
&\qquad\wedge f\,(\tau,d\bar{z})+\\
&+\sum_{q\neq j}(-1)^{j-1}\,\mathcal{U}_q\,(\tau,z)\,d\bar{\tau}_{[j]}\wedge d\tau\wedge\widehat{d\bar{z}^q}\wedge d\bar{z}^j\wedge f\,(\tau,d\bar{z})+\\
&+(-1)^m\sum_{q\neq j}(-1)^{j-1}\,\overline{\mathcal{U}}_q\,(\tau,z)\,d\bar{\tau}\wedge d\tau_{[j]}\wedge d\bar{z}^q\wedge\widehat{d\bar{z}^j}\wedge\\
&\wedge f\cdot(\tau,d\bar{z})\Big\}(2i)^m\int\limits_{\Omega^+}\Big\{\sum_{q=1}^{m}\mathcal{U}_q\,(\tau,z)\,\widehat{d\bar{z}^q}\wedge\bar{\partial}f\,(\tau,d\bar{z})+\\
&+\sum_{q=1}^{m}\overline{\mathcal{U}}_q\,(\tau,z)\,d\bar{z}^q\wedge\bar{\partial}^*f\,(\tau,d\bar{z})\Big\}dV_\tau;
\end{aligned}
\tag{8.14.3}
$$

$$
\begin{aligned}
0\;=\;&\int\limits_{\Gamma}\Big\{\sum_{q<j}\Big((-1)^{j-1}\overline{\mathcal{U}}_q(\tau,\,z)d\bar{\tau}_{[j]}\wedge d\tau\;-\\
&-(-1)^{q-1}\overline{\mathcal{U}}_j(\tau,z)d\bar{\tau}_{[q]}\wedge d\tau\Big)\wedge d\bar{z}^q\wedge d\bar{z}^j\wedge f(\tau,d\bar{z})+\\
&+(-1)^m\sum_{q<j}\Big((-1)^{j-1}\mathcal{U}_q(\tau,z)d\bar{\tau}\wedge d\tau_{[j]}\;-\\
&-(-1)^{q-1}\mathcal{U}_j(\tau,z)d\bar{\tau}\wedge d\tau_{[q]}\Big)\wedge\widehat{d\bar{z}^q}\wedge\widehat{d\bar{z}^j}\wedge f(\tau,d\bar{z})\Big\}-\\
&-(2i)^m\int\limits_{\Omega^+}\Big\{\sum_{q=1}^{m}\overline{\mathcal{U}}_q(\tau,z)d\bar{z}^q\wedge\overline{\partial}f(\tau,d\bar{z})+\\
&+\sum_{q=1}^{m}\mathcal{U}_q(\tau,z)\widehat{d\bar{z}^q}\wedge\bar{\partial}^*f(\tau,d\bar{z})\Big\}dV_\tau.
\end{aligned}
\tag{8.14.4}
$$

We are going to compare the formulas (8.14.3) and (8.14.4) with

the well-known Koppelman (or Bochner-Martinelli-Koppelman) formula, see, e.g., §1 in [Ky], [HL1], Chap. I in [HL2], Chap. IV in [R]. To this end we start with the following assertion.

8.15 Partial derivatives of integrals with a weak singularity

Theorem *Let Ω be a bounded domain in $\mathbb{C}^m$ with the piecewise smooth boundary Γ, let $g \in C^1(\overline{\Omega};\mathbb{C})$, then for any $k, j \in \{1,\dots,m\}$ there holds*

$$\begin{aligned}\frac{\partial}{\partial\overline{z}_k}\int\limits_{\Omega} g(\tau)\mathcal{U}_j(\tau,z)d\overline{\tau}\wedge d\tau = \\ = \ (-1)^{m-j}\int\limits_{\Gamma} g(\tau)\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau_{[j]} + \int\limits_{\Omega}\frac{\partial g}{\partial\tau_j}\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau;\end{aligned} \tag{8.15.1}$$

$$\begin{aligned}\frac{\partial}{\partial z_k}\int\limits_{\Omega} g(\tau)\overline{\mathcal{U}}_j(\tau,z)d\overline{\tau}\wedge d\tau = \\ = \ (-1)^{j}\int\limits_{\Gamma} g(\tau)\mathcal{U}_k(\tau,z)d\tau\wedge d\overline{\tau}_{[j]} + \int\limits_{\Omega}\frac{\partial g}{\partial\overline{\tau}_j}\mathcal{U}_k(\tau,z)d\overline{\tau}\wedge d\tau,\end{aligned} \tag{8.15.2}$$

where

$$\begin{aligned}\mathcal{U}_j(\tau,z) &= \frac{(m-1)!}{(2\pi i)^m},\frac{\overline{\tau}_j-\overline{z}_j}{|\tau-z|^{2m}};\\ \overline{\mathcal{U}}_j(\tau,z) &= \frac{(m-1)!}{(2\pi i)^m},\frac{\tau_j-z_j}{|\tau-z|^{2m}}.\end{aligned}$$

Proof. It is enough to prove the equality (8.15.1) only. Given $j \in \{1,\dots,m\}$, consider the differential form

$$\omega_j := (-1)^{m+j-1}\cdot g(\tau)|\tau-z|^{2-2m}d\overline{\tau}\wedge d\tau_{[j]},$$

for which we have

$$d\omega_j = \sum_{k=1}^{m}\frac{\partial}{\partial\overline{\tau}_k}\left[(-1)^{m+j-1}g(\tau)\cdot|\tau-z|^{2-2m}\right]d\overline{\tau}_k\wedge d\overline{\tau}\wedge d\tau_{[j]} +$$

$$+\sum_{k=1}^{m}\frac{\partial}{\partial\tau_k}\left[(-1)^{m+j-1}g(\tau)\cdot|\tau-z|^{2-2m}\right]d\tau_k\wedge d\overline{\tau}\wedge d\tau_{[j]}=$$

$$= \quad (-1)^{m+j-1}\frac{\partial}{\partial\tau_j}\left(g(\tau)|\tau-z|^{2-2m}\right)d\tau_j\wedge d\overline{\tau}\wedge d\tau_{[j]}=$$

$$= \quad \left(\frac{\partial g}{\partial\tau_j}\cdot|\tau-z|^{2-2m}+g(\tau)\cdot\frac{\partial}{\partial\tau_j}\left(|\tau-z|^{2-2m}\right)\right)d\overline{\tau}\wedge d\tau.$$

But

$$\frac{\partial}{\partial\tau_j}\left(|\tau-z|^{2-2m}\right)=(1-m)\frac{\tau_j-z_j}{|\tau-z|^{2m}},$$

hence

$$d\omega_j=\left(\frac{\partial g}{\partial\tau_j}\cdot|\tau-z|^{2-2m}+(1-m)g(\tau)\frac{\overline{\tau}_j-\overline{z}_j}{|\tau-z|^{2m}}\right)d\overline{\tau}\wedge d\tau.$$

Both ω on Γ and $d\omega$ on Ω are absolutely integrable, thus the Stokes formula is valid:

$$(-1)^{m-j+1}\int_{\Gamma}g(\tau)\cdot|\tau-z|^{2-2m}d\overline{\tau}\wedge d\tau_{[j]}=$$

$$= \quad \int_{\Omega}\frac{\partial g}{\partial\tau_j}\cdot|\tau-z|^{2-2m}d\overline{\tau}\wedge d\tau+(1-m)\int_{\Omega}g(\tau)\frac{\overline{\tau}_j-\overline{z}_j}{|\tau-z|^{2m}}d\overline{\tau}\wedge d\tau,$$

which implies that

$$\int_{\Omega}g(\tau)\frac{\overline{\tau}_j-\overline{z}_j}{|\tau-z|^{2m}}d\overline{\tau}\wedge d\tau=$$

$$= \quad \frac{(-1)^{m-j+1}}{1-m}\int_{\Gamma}g(\tau)\cdot|\tau-z|^{2-2m}d\overline{\tau}\wedge d\tau_{[j]}-$$

$$-\int_{\Omega}\frac{\partial g}{\partial\tau_j}|\tau-z|^{2-2m}d\overline{\tau}\wedge d\tau=:I_1+I_2.$$

For $z\in\Omega,\ I_1$ is differentiable and by Leibnitz's rule

$$\frac{\partial I_1}{\partial\overline{z}_k}=(-1)^{m-j}\int_{\Gamma}g(\tau)\frac{\tau_k-z_k}{|\tau-z|^{2m}}d\overline{\tau}\wedge d\tau_{[j]}.$$

The integral I_2 is improper; differentiating under the integral sign we arrive again at the improper integral both being absolutely convergent, thus

$$\frac{\partial I_2}{\partial \overline{z}_k} = \int\limits_{\Omega} \frac{\tau_k - z_k}{|\tau - z|^{2m}} \frac{\partial g}{\partial \tau_j} d\overline{\tau} \wedge d\tau.$$

Hence we get, finally, the equality

$$\begin{aligned} \frac{\partial}{\partial \overline{z}_k} \int\limits_{\Omega} g(\tau) \frac{\overline{\tau}_j - \overline{z}_j}{|\tau - z|^{2m}} d\overline{\tau} \wedge d\tau = \\ = \quad (-1)^{m-j} \int\limits_{\Gamma} g(\tau) \frac{\tau_k - z_k}{|\tau - z|^{2m}} d\overline{\tau} \wedge d\tau_{[j]} \\ + \int\limits_{\Omega} \frac{\partial g}{\partial \tau_j} \cdot \frac{\tau_k - z_k}{|\tau - z|^{2m}} \cdot d\overline{\tau} \wedge d\tau, \end{aligned}$$

which is exactly the equality (8.15.1). One can compare all this with Lemma 1.15 from [Ky].

8.16 Theorem 8.15 in $\mathbb{C}^2$

Having in mind further development, in this subsection we write down explicitly what Theorem 8.15 gives for $m = 2$. We obtain the following four formulas from (8.15.1):

if $k = 1, j = 1$ then

$$\begin{aligned} \frac{\partial}{\partial \overline{z}_1} \int\limits_{\Omega} g(\tau) \mathcal{U}_1(\tau, z) d\overline{\tau} \wedge d\tau = \\ = \quad - \int\limits_{\Gamma} g(\tau) \overline{\mathcal{U}}_1(\tau, z) d\overline{\tau} \wedge d\tau_2 + \int\limits_{\Omega} \frac{\partial f}{\partial \tau_1} \overline{\mathcal{U}}_1(\tau, z) d\overline{\tau} \wedge d\tau; \end{aligned} \tag{8.16.1}$$

if $k = 1, j = 2$ then

$$\frac{\partial}{\partial \overline{z}_1} \int\limits_{\Omega} g(\tau) \mathcal{U}_2(\tau, z) d\overline{\tau} \wedge d\tau =$$

$$= \int_\Gamma g(\tau)\overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau_1 + \int_\Omega \frac{\partial g}{\partial \tau_2}\cdot\overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau; \tag{8.16.2}$$

if $k=2, j=1$ then

$$\frac{\partial}{\partial \overline{z}_2}\int_\Omega g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau =$$
$$= -\int_\Gamma g(\tau)\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau_2 + \int_\Omega \frac{\partial g}{\partial \tau_1}\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau; \tag{8.16.3}$$

if $k=2, j=2$ then

$$\frac{\partial}{\partial \overline{z}_2}\int_\Omega g(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau =$$
$$= \int_\Gamma g(\tau)\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau_1 + \int_\Omega \frac{\partial g}{\partial \tau_2}\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau. \tag{8.16.4}$$

8.17 Formula (8.14.3) in $\mathbb{C}^2$

To illustrate the general idea, we begin with comparing the formulas (8.14.3) and (8.14.4) and the Koppelman formula in $\mathbb{C}^2$. The interesting case here is that of a differential form of degree 1, i.e., let $m=2, f = f_1(\tau)d\overline{z}_1 + f_2(\tau)d\overline{z}_2$. Substituting this into (8.14.3) we obtain the equality

$$f_1 d\overline{z}_1 + f_2 d\overline{z}_2 =$$
$$= d\overline{z}_1\int_\Gamma\Big(f_1(\tau)(\mathcal{U}_2(\tau,z)d\overline{\tau}_1\wedge d\tau - \mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau_2) +$$
$$+ f_2(\tau)\left(\mathcal{U}_2(\tau,z)d\overline{\tau}_2\wedge d\tau + \overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau_1\right)\Big) +$$
$$+ d\overline{z}_2\int_\Gamma\Big(f_2(\tau)\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau_1 - \mathcal{U}_1(\tau,z)d\overline{\tau}_2\wedge d\tau\Big) -$$

$$
\begin{aligned}
&- f_1(\tau)\cdot\left(\mathcal{U}_1(\tau,z)d\overline{\tau}_1\wedge d\tau+\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau_2\right)+\\
&+d\overline{z}_1\int\limits_{\Omega}\left((\frac{\partial f_1}{\partial z_1}+\frac{\partial f_2}{\partial z_2})\overline{\mathcal{U}}_1(\tau,z)-\right.\\
&\qquad\left.-(\frac{\partial f_2}{\partial\overline{z}_1}-\frac{\partial f_1}{\partial\overline{z}_2})\mathcal{U}_2(\tau,z)\right)d\overline{\tau}\wedge d\tau+\\
&+d\overline{z}_2\int\limits_{\Omega}\left((\frac{\partial f_1}{\partial z_1}+\frac{\partial f_2}{\partial z_2})\overline{\mathcal{U}}_2(\tau,z)-\right.\\
&\qquad\left.-(\frac{\partial f_2}{\partial\overline{z}_1}-\frac{\partial f_1}{\partial\overline{z}_2})\mathcal{U}_1(\tau,z)\right)d\overline{\tau}\wedge d\tau.
\end{aligned}
\tag{8.17.1}
$$

Both functions f_1 and f_2 are arbitrary; in particular, one of them may be taken identically zero, which leads to the equalities

$$
\begin{aligned}
f_1(z) &= \int\limits_{\Gamma} f_1(\tau)\left[\mathcal{U}_2(\tau,z)d\overline{\tau}_1\wedge d\tau-\overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau_2\right]+\\
&\quad+\int\limits_{\Omega}\left(\frac{\partial f_1}{\partial z_1}\overline{\mathcal{U}}_1(\tau,z)+\frac{\partial f_1}{\partial\overline{z}_2}\mathcal{U}_2(\tau,z)\right)d\overline{\tau}\wedge d\tau; \qquad (8.17.2)\\
0 &= \int\limits_{\Gamma} f_1(z)(\mathcal{U}_1(\tau,z)d\overline{\tau}_1\wedge d\tau+\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau_2)-\\
&\quad-\int\limits_{\Omega}\left(\frac{\partial f_1}{\partial z_1}\overline{\mathcal{U}}_2(\tau,z)-\frac{\partial f_1}{\partial\overline{z}_2}\mathcal{U}_1(\tau,z)\right)(\tau,z)d\overline{\tau}\wedge d\tau; \qquad (8.17.3)\\
f_2(z) &= \int\limits_{\Gamma} f_2(\tau)\left[\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau_1-\mathcal{U}_1(\tau,z)d\overline{\tau}_2\wedge d\tau\right]+\\
&\quad+\int\limits_{\Omega}\left(\frac{\partial f_2}{\partial z_2}\overline{\mathcal{U}}_2(\tau,z)+\frac{\partial f_2}{\partial\overline{z}_1}\mathcal{U}_1(\tau,z)\right)d\overline{\tau}\wedge d\tau; \qquad (8.17.4)\\
0 &= \int\limits_{\Gamma} f_2(\tau)\left[\mathcal{U}_2(\tau,z)d\overline{\tau}_2\wedge d\tau+\overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau_1\right]+
\end{aligned}
$$

$$+\int\limits_{\Omega}\left(\frac{\partial f_2}{\partial z_2}\overline{\mathcal{U}}_1(\tau,z)-\frac{\partial f_2}{\partial\overline{z}_1}\mathcal{U}_2(\tau,z)\right)d\overline{\tau}\wedge d\tau. \tag{8.17.5}$$

What is more, f_1 and f_2 may be taken real-valued which means that we have, in fact, two pairs of formulas such that in each pair one of the formulas is the complex conjugate to the other one. We resume the above reasonings as follows.

8.18 Integral representation (8.14.3) for a $(0,1)$-differential form in $\mathbb{C}^2$, in terms of its coefficients

Proposition *Let $\Omega\subset\mathbb{C}^2$ and $\Gamma=\partial\Omega$ be as always, and let $g\in C^1(\overline{\Omega};\mathbb{C})$. Then for any $z\in\Omega$ there holds*

$$\begin{aligned} g(z) &= \int\limits_{\Gamma} g(\tau)\left(\mathcal{U}_2(\tau;z)\cdot d\overline{\tau}_1\wedge d\tau-\overline{\mathcal{U}}_1(\tau;z)d\overline{\tau}\wedge d\tau_2\right)+\\ &\quad+\int\limits_{\Omega}\left(\frac{\partial g}{\partial z_1}\overline{\mathcal{U}}_1(\tau;z)+\frac{\partial g}{\partial\overline{z}_2}\mathcal{U}_2(\tau;z)\right)d\overline{\tau}\wedge d\tau; \end{aligned} \tag{8.18.1}$$

$$\begin{aligned} 0 &= \int\limits_{\Gamma} g\,(\tau)(\mathcal{U}_1(\tau;z)\cdot d\overline{\tau}_1\wedge d\tau+\overline{\mathcal{U}}_2(\tau;z)d\overline{\tau}\wedge d\tau_2)-\\ &\quad-\int\limits_{\Omega}\left(\frac{\partial g}{\partial z_1}\overline{\mathcal{U}}_2(\tau;z)-\frac{\partial g}{\partial\overline{z}_2}\mathcal{U}_1(\tau;z)\right)d\overline{\tau}\wedge d\tau. \end{aligned} \tag{8.18.2}$$

8.19 Koppelman's formula in $\mathbb{C}^2$

Take again the differential form $f=f_1d\overline{z}_1+f_2d\overline{z}_2$ (in $\mathbb{C}^2$) and substitute it into the Koppelman formula. We obtain

$$f_1d\overline{z}_1+f_2d\overline{z}_2=$$

$$
\begin{aligned}
= \; & d\overline{z}_1 \int_\Gamma (f_1(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}_1 \wedge d\tau + f_2\mathcal{U}_2(\tau,z)d\overline{\tau}_2 \wedge d\tau) - \\
& - d\overline{z}_2 \int_\Gamma (f_1\mathcal{U}_1(\tau,z)d\overline{\tau}_1 \wedge d\tau + f_2\mathcal{U}_1(\tau,z)d\overline{\tau}_2 \wedge d\tau) + \\
& + d\overline{z}_1 \cdot \frac{\partial}{\partial \overline{z}_1} \int_\Omega (f_1\mathcal{U}_1(\tau,z) + f_2\mathcal{U}_2(\tau,z))\, d\overline{\tau} \wedge d\tau + \\
& + d\overline{z}_2 \cdot \frac{\partial}{\partial \overline{z}_2} \int_\Omega (f_1\mathcal{U}_1(\tau,z) + f_2\mathcal{U}_2(\tau,z))\, d\overline{\tau} \wedge d\tau - \\
& - d\overline{z}_1 \int_\Omega \left(\frac{\partial f_2}{\partial \overline{\tau}_1} - \frac{\partial f_1}{\partial \overline{\tau}_2}\right) \mathcal{U}_2(\tau,z) d\overline{\tau} \wedge d\tau + \\
& + d\overline{z}_2 \int_\Omega \left(\frac{\partial f_2}{\partial \overline{\tau}_1} - \frac{\partial f_1}{\partial \overline{\tau}_2}\right) \mathcal{U}_1(\tau,z) d\overline{\tau} \wedge d\tau. \qquad (8.19.1)
\end{aligned}
$$

We now apply the same procedure as in Subsection 8.17: separate the coordinates of $d\overline{z}_1$ and $d\overline{z}_2$ and take into account that f_1 and f_2 are arbitrary smooth functions. This gives the following assertion.

8.20 Koppelman's formula in $\mathbb{C}^2$ for a $(0,1)$ - differential form, in terms of its coefficients

Proposition *Let $\Omega \subset \mathbb{C}^2$ and $\Gamma = \partial\Omega$ be as always, and let $g \in C^1(\overline{\Omega};\mathbb{C})$. Then for any $z \in \Omega$ there holds*

$$
\begin{aligned}
g(z) \; = \; & \int_\Gamma g(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}_1 \wedge d\tau + \\
& + \frac{\partial}{\partial \overline{z}_1} \int_\Omega g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau} \wedge d\tau + \int_\Omega \frac{\partial g}{\partial \overline{\tau}_2}\mathcal{U}_2(\tau,z)d\overline{\tau} \wedge d\tau;
\end{aligned}
\qquad (8.20.1)
$$

$$
g(z) \; = \; - \int_\Gamma g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}_2 \wedge d\tau +
$$

$$+\frac{\partial}{\partial\overline{z}_2}\int_\Omega g(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_1}\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau; \tag{8.20.2}$$

$$\begin{aligned}0 &= \int_\Gamma g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}_1\wedge d\tau-\\ &-\frac{\partial}{\partial\overline{z}_2}\int_\Omega g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau;\end{aligned} \tag{8.20.3}$$

$$\begin{aligned}0 &= \int_\Gamma g(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}_2\wedge d\tau+\\ &+\frac{\partial}{\partial\overline{z}_1}\int_\Omega g(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau-\int_\Omega\frac{\partial g}{\partial\overline{\tau}_1}\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau.\end{aligned} \tag{8.20.4}$$

8.21 Comparison of Propositions 8.18 and 8.20

Now we are ready to compare the formulas in Proposition 8.20 and Proposition 8.18. Take the formula (8.20.1) and apply the equality (8.16.1):

$$\begin{aligned}g(z) &= \int_\Gamma g(\mathcal{U}_2(\tau,z)d\overline{\tau}_1\wedge d\tau+\frac{\partial}{\partial\overline{z}_1}\int_\Omega g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau\\ &+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_1(\tau,z),\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau=\\ &= \int_\Gamma g(\tau)\mathcal{U}_2(\tau,z)d\overline{\tau}_1\wedge d\tau-\int_\Gamma g(\tau)\overline{\mathcal{U}}_1(\tau,z)d\tau_2\wedge d\overline{\tau}+\\ &+\int_\Omega\frac{\partial g}{\partial\tau_1}\overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau=\\ &= \int_\Gamma g(\tau)(\mathcal{U}_2(\tau,z)d\overline{\tau}_1\wedge d\tau-\overline{\mathcal{U}}_1(\tau,z)d\tau_2\wedge d\overline{\tau})+\\ &+\int_\Omega\left(\frac{\partial g}{\partial\tau_1}\overline{\mathcal{U}}_1(\tau,z)+\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_2(\tau,z)\right)d\overline{\tau}\wedge d\tau,\end{aligned}$$

which is exactly the formula (8.18.2). Take now (8.20.1) and apply (8.16.4) to the term with $\dfrac{\partial}{\partial\overline{z}_2}$; we get

$$\begin{aligned} g(z) &= -\int_\Gamma g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}_2\wedge d\tau+\int_\Gamma g(\tau)\overline{\mathcal{U}}_2(\tau,z)d\tau_1\wedge d\overline{\tau}+\\ &\quad+\int_\Omega\frac{\partial g}{\partial\tau_2}\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_1}\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau=\\ &= \int_\Gamma g(\tau)\left(\overline{\mathcal{U}}_2(\tau,z)d\tau_1\wedge d\overline{\tau}-\mathcal{U}_1(\tau,z)d\overline{\tau}_2\wedge d\tau\right)+\\ &\quad+\int_\Omega\left(\frac{\partial g}{\partial\tau_2}\overline{\mathcal{U}}_2(\tau,z)+\frac{\partial g}{\partial\overline{\tau}_1}\mathcal{U}_1(\tau,z)\right)d\overline{\tau}\wedge d\tau. \end{aligned}$$

The function f is an arbitrary function, in particular, a real-valued one, hence this is just the complex conjugate to the formula (8.18.1).

For the formula (8.20.1) we obtain, using (8.16.3)

$$\begin{aligned} 0 &= \int_\Gamma g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}_1\wedge d\tau-\frac{\partial}{\partial\overline{z}_2}\int_\Omega g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau+\\ &\quad+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau=\\ &= \int_\Gamma g(\tau)\mathcal{U}_1(\tau,z)d\overline{\tau}_1\wedge d\tau+\int_\Gamma g(\tau)\overline{\mathcal{U}}_2(\tau,z)d\tau_2\wedge d\overline{\tau}-\\ &\quad-\int_\Omega\frac{\partial g}{\partial\tau_1}\overline{\mathcal{U}}_2(\tau,z)d\overline{\tau}\wedge d\tau+\int_\Omega\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_1(\tau,z)d\overline{\tau}\wedge d\tau=\\ &= \int_\Gamma g(\tau)(\mathcal{U}_1(\tau,z)d\overline{\tau}_1\wedge d\tau+\overline{\mathcal{U}}_2(\tau,z)d\tau_2\wedge d\overline{\tau})+\\ &\quad+\int_\Omega\left(\frac{\partial g}{\partial\overline{\tau}_2}\mathcal{U}_1(\tau,z)-\frac{\partial g}{\partial\tau_1}\overline{\mathcal{U}}_2(\tau,z)\right)d\overline{\tau}\wedge d\tau, \end{aligned}$$

which is the formula (8.18.2), while (8.20.1) gives its complex conjugate if one makes use of (8.16.2):

$$\begin{aligned} 0 &= \int_\Gamma g\mathcal{U}_2(\tau,z)d\overline{\tau}_2\wedge d\tau+\frac{\partial}{\partial\overline{z}_1}\int_\Omega g\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau-\\ &\quad-\int_\Omega\frac{\partial g}{\partial\overline{\tau}_1}\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau=\\ &= \int_\Gamma g\mathcal{U}_2(\tau,z)d\overline{\tau}_2\wedge d\tau+\int_\Gamma g\overline{\mathcal{U}}_1(\tau,z)d\tau_1\wedge d\overline{\tau}+ \end{aligned}$$

$$+\int_\Omega \frac{\partial g}{\partial \tau_2}\overline{\mathcal{U}}_1(\tau,z)d\overline{\tau}\wedge d\tau-\int_\Omega \frac{\partial g}{\partial \overline{\tau}_1}\mathcal{U}_2(\tau,z)d\overline{\tau}\wedge d\tau=$$

$$=\int_\Gamma g(\tau)\left(\mathcal{U}_2(\tau,z)d\overline{\tau}_2\wedge d\tau+\overline{\mathcal{U}}_1(\tau,z)d\tau_1\wedge d\overline{\tau}\right)+$$

$$+\int_\Omega\left(\frac{\partial g}{\partial \overline{\tau}_2}\overline{\mathcal{U}}_1(\tau,z)-\frac{\partial g}{\partial \overline{\tau}_1}\mathcal{U}_2(\tau,z)\right)d\overline{\tau}\wedge d\tau.$$

8.22 Koppelman's formula in $\mathbb{C}^2$ and hyperholomorphic theory

Thus we may conclude already that, at least in $\mathbb{C}^2$, the Koppelman formula is another way of writing the integral representation (8.14.3). We see the following advantage of the formula (8.14.3): it reveals the deep and intimate relation between the Koppelman formula and the complex Hodge-Dolbeault system for which (8.14.3) is its "Borel-Pompeiu-type" integral representation. It shows also that a "Cauchy-type" integral representation for the Koppelman formula cannot be obtained by just crossing out both volume integrals; one must first perform a transformation of the kind described by Theorem 8.15, thus arriving at (8.14.3) and seeing that the volume integrals disappear for a differential form which is a solution of the complex Hodge-Dolbeault system. In this way the relation between the Koppelman formula and the hyperholomorphic theory is established as well. This relation involves the equation (8.14.4) also: the Koppelman formula for two variables (8.19.1) written in the form (8.17.1) and taken together with the corresponding corollary of (8.3) gives the hyperholomorphic Borel-Pompeiu formula for differential forms.

8.23 Definition of $\rho_{H,K}$

Certain calculations in what follows get simplified if one uses the notion we shall introduce now. Let H and K be arbitrary finite sets of integers with cardinal numbers $card\ H$ and $card\ K$, respectively. Then

$$\rho_{H,K}:=\begin{cases}0, & \text{if} \quad H\cap K\neq\emptyset,\\ (-1)^{\kappa}, & \text{if} \quad H\cap K=\emptyset,\end{cases}$$

where κ is the number of elements in the Cartesian product $H \times K = \{(h,k)\,|h \in H, k \in K\}$ with the property $h > k$. Those elements are called sometimes the **inversions** of the pair (H,K).

One can prove that, in particular, for the three sets H, K, L there hold

$$\star \quad \rho_{H,K} \cdot \rho_{H \cup K,L} = \rho_{K,L} \cdot \rho_{H,K \cup L} = \rho_{H,K} \cdot \rho_{K,L} \cdot \rho_{H,L}; \tag{8.23.1}$$

$$\star \quad \rho_{H,K} = (-1)^{cardH \cdot cardK}\, \rho_{K,H}; \tag{8.23.2}$$

$$\star \quad \rho_{H,K \cup L} = \rho_{H,K} \cdot \rho_{H,L}. \tag{8.23.3}$$

More properties and explanations will be given in Subsection 8.29.

We shall make use of these properties working with wedge products of differential forms and with the operator $\widehat{d\overline{z}^i}$. For example, let H and K be ordered subsets of $\{1,...,m\}$ and let $H \cup K$ denote the **ordered** union of the two sets, then

$$d\bar{z}^H \wedge d\bar{z}^K = \rho_{H,K} d\bar{z}^{H \cup K}.$$

If $j \in \{1,...,m\}$ then

$$\widehat{d\bar{z}^j} \wedge d\bar{z}^H = \rho_{\{j\},H \setminus \{j\}} \cdot d\bar{z}^{H \setminus \{j\}}.$$

8.24 A reformulation of the Borel-Pompeiu formula

Let us proceed now to the general case of m variables. It is enough to consider a differential form $f = f_J d\overline{z}^J$ with $|J| = q$ fixed ($q \in \{0,1,\ldots,m\}$), $J = (j_1,\ldots,j_q)$, $j_1 < j_2 < \ldots < j_q$.

Then the formula (8.14.3) reads

$$f(z) \;=\; (-1)^{\frac{m(m-1)}{2}} \int_\Gamma \Big\{ \sum_{i \notin J} (-1)^{i-1} f_J(\tau) \mathcal{U}_i(\tau,z) d\overline{\tau}_{[i]} \wedge d\tau \wedge d\overline{z}^J +$$

$$
\begin{aligned}
&+ (-1)^m \sum_{i\in J}(-1)^{i-1} f_J(\tau)\overline{\mathcal{U}}_i(\tau,z)d\overline{\tau}\wedge d\tau_{[i]}\wedge d\overline{z}^J + \\
&+ \sum_{i\notin J}\sum_{p\neq i}(-1)^{i-1} f_J(\tau)\mathcal{U}_p(\tau,z)\cdot \\
&\qquad \cdot d\overline{\tau}_{[i]}\wedge d\tau\wedge\widehat{d\overline{z}^p}\wedge d\overline{z}^i\wedge d\overline{z}^J + \\
&+ \sum_{i\in J}\sum_{p\neq i}(-1)^{m+i-1} f_J(\tau)\overline{\mathcal{U}}_p(\tau,z)\cdot \\
&\qquad \cdot d\overline{\tau}\wedge d\tau_{[i]}\wedge d\overline{z}^p\wedge\widehat{d\overline{z}^i}\wedge d\overline{z}^J\Big\} - \\
&- (-1)^{\frac{m(m-1)}{2}}\int_\Omega\Big\{\sum_{p=1}^m \mathcal{U}_p(\tau,z)\sum_{n\notin J}\frac{\partial f_J}{\partial\overline{z}_n}\widehat{d\overline{z}^p}\wedge d\overline{z}^n\wedge d\overline{z}^J + \\
&+ \sum_{p=1}^m \overline{\mathcal{U}}_p(\tau,z)\sum_{n\in J}\frac{\partial f}{\partial z_n}d\overline{z}^p\wedge\widehat{d\overline{z}}_n\wedge d\overline{z}^J\Big\}d\overline{\tau}\wedge d\tau. \qquad (8.24.1)
\end{aligned}
$$

The first two surface integrals are already of the form we need. For the third surface integral we have

$$
\begin{aligned}
&\int_\Gamma \sum_{\substack{p\neq i\\ i\notin J}}(-1)^{i-1} f_J(\tau)\mathcal{U}_p(\tau,z)d\overline{\tau}_{[i]}\wedge d\tau\wedge\widehat{d\overline{z}^p}\wedge\rho_{\{i\},J}d\overline{z}^{J\cup\{i\}} = \\
&= \int_\Gamma \sum_{\substack{i\notin J\\ p\in J}}(-1)^{i-1} f_J(\tau)\mathcal{U}_p(\tau,z)\rho_{\{i\},J}\cdot\rho_{\{p\},(J\cup\{i\})\setminus\{p\}}\cdot \\
&\qquad \cdot d\overline{\tau}_{[i]}\wedge d\tau\wedge d\overline{z}^{(J\cup\{i\})\setminus\{p\}}.
\end{aligned}
$$

Analogously, for the fourth surface integral we get

$$
\begin{aligned}
&\int_\Gamma \sum_{\substack{i\in J\\ p\neq i}}(-1)^{m+i-1} f_J(\tau)\overline{\mathcal{U}}_p(\tau,z)\cdot\rho_{\{i\},J\setminus\{i\}}\cdot \\
&\qquad \cdot d\overline{\tau}\wedge d\tau_{[i]}\wedge d\overline{z}^p\wedge d\overline{z}^{J\setminus\{i\}} = \\
&= \int_\Gamma \sum_{\substack{i\in J\\ p\notin J}}(-1)^{m+i-1} f_J(\tau)\overline{\mathcal{U}}_p(\tau,z)\rho_{\{i\},J\setminus\{i\}}\cdot\rho_{\{p\},J\setminus\{i\}}\cdot \\
&\qquad \cdot d\overline{\tau}\wedge d\tau_{[i]}\wedge d\overline{z}^{(J\setminus\{i\})\cup\{p\}}.
\end{aligned}
$$

Now we proceed to the volume integrals in (8.24.1). For the first of them we get

$$\int_\Omega \sum_{n\notin J}\sum_{p=1}^{m} \frac{\partial f_J}{\partial \overline{z}_n}(\tau)\cdot \mathcal{U}_p(\tau,z)\rho_{\{n\},J}d\overline{\tau}\wedge d\tau\wedge$$

$$\wedge \widehat{d\overline{z}^p}\wedge d\overline{z}^{J\cup\{n\}} =$$

$$\begin{aligned}
&= \int_\Omega \sum_{n\notin J}\Big(\sum_{p=n}+\sum_{p\in J}+\sum_{p\notin J\cup\{n\}}\Big)\cdot \\
&\quad \cdot\frac{\partial f_J}{\partial \overline{z}_n}(\tau)\mathcal{U}_p(\tau,z)\rho_{\{n\},J}\cdot d\overline{\tau}\wedge d\tau\wedge\widehat{d\overline{z}^p}\wedge d\overline{z}^{J\cup\{n\}} = \\
&= \int_\Omega \sum_{p\notin J}\frac{\partial f_J}{\partial \overline{z}_p}(\tau)\cdot \mathcal{U}_p(\tau,z)d\overline{\tau}\wedge d\tau\cdot\rho_{\{p\},J}\cdot\rho_{\{p\},J}\cdot d\overline{z}^J + \\
&\quad + \int_\Omega \sum_{n\notin J}\sum_{p\in J}\frac{\partial f_J}{\partial \overline{z}_n}(\tau)\cdot \mathcal{U}_p(\tau,z)d\overline{\tau}\wedge d\tau\rho_{\{n\},J}\ \cdot \\
&\quad \cdot\rho_{\{p\},(J\cup\{n\})\setminus\{p\}}\cdot d\overline{z}^{(J\cup\{n\})\setminus\{p\}}.
\end{aligned}$$

Analogously, for the last volume integral we obtain

$$\begin{aligned}
&\int\limits_\Omega \sum_{n\in J}\sum_{p=1}^{m}\frac{\partial f_J}{\partial z_n}(\tau)\overline{\mathcal{U}}_p(\tau,z)\cdot\rho_{\{n\},J\setminus\{n\}}d\overline{\tau}\wedge d\tau\wedge d\overline{z}^p\wedge d\overline{z}^{J\setminus\{n\}} = \\
&= \int\limits_\Omega \sum_{n\in J}\Big(\sum_{p=n}+\sum_{p\in J\setminus\{n\}}+\sum_{p\notin J}\Big)\cdot \\
&\qquad \cdot\frac{\partial f_J}{\partial z_n}\overline{\mathcal{U}}_p(\tau,z)\cdot\rho_{\{n\},J\setminus\{n\}}d\overline{\tau}\wedge d\tau\wedge d\overline{z}^p\wedge d\overline{z}^{J\setminus\{n\}} = \\
&= \int\limits_\Omega \sum_{n\in J}\frac{\partial f_J}{\partial z_n}(\tau)\cdot\overline{\mathcal{U}}_n(\tau,z)d\overline{\tau}\wedge d\tau\wedge d\overline{z}^J + \\
&\quad + \int\limits_\Omega \sum_{\substack{n\in J\\ p\notin J}}\frac{\partial f_J}{\partial z_n}\overline{\mathcal{U}}_p(\tau,z)\cdot\rho_{\{n\},J\setminus\{n\}}\cdot \\
&\qquad \cdot\rho_{\{p\},J\setminus\{n\}}d\overline{\tau}\wedge d\tau\wedge d\overline{z}^{(J\setminus\{n\})\cup\{p\}}.
\end{aligned}$$

Substituting all this into (8.24.1) and comparing the differentials in both sides, we conclude that (8.24.1) is equivalent to the following

system:

$$
\begin{aligned}
f_J(z) \;=\; & (-1)^{\frac{m(m-1)}{2}} \Bigg\{ \int_\Gamma \Bigg(\sum_{i\notin J} (-1)^{i-1} f_J(\tau)\mathcal{U}_i(\tau,z) d\overline{\tau}_{[i]} \wedge d\tau + \\
& + (-1)^m \sum_{i\in J} (-1)^{i-1} f_J(\tau)\overline{\mathcal{U}}_i(\tau,z) d\overline{\tau}\wedge d\tau_{[i]} \Bigg) - \\
& - \int_\Omega \sum_{p\notin J} \frac{\partial f_J}{\partial \overline{z}_p}(\tau)\cdot \mathcal{U}_p(\tau,z) d\overline{\tau}\wedge d\tau - \\
& - \int_\Omega \sum_{n\in J} \frac{\partial f_J}{\partial z_n}(\tau)\overline{\mathcal{U}}_n(\tau,z)\cdot d\overline{\tau}\wedge d\tau \Bigg\},
\end{aligned}
\tag{8.24.2}
$$

$$
\begin{aligned}
0 \;=\; & \int_\Gamma \big[(-1)^{i-1} f_J(\tau)\mathcal{U}_p(\tau,z)\rho_{\{i\},J}\cdot \rho_{\{p\},(J\cup\{i\})\setminus\{p\}} d\overline{\tau}_{[i]}\wedge d\tau + \\
& + (-1)^{m+p-1} f_J(\tau)\overline{\mathcal{U}}_i(\tau,z)\rho_{\{p\},J\setminus\{p\}}\cdot \rho_{\{i\},J\setminus\{p\}} d\overline{\tau}\wedge d\tau_{[p]} \big] - \\
& - \int_\Omega \Bigg[\frac{\partial f_J}{\partial \overline{z}_i}(\tau)\mathcal{U}_p(\tau,z)\rho_{\{i\},J}\cdot \rho_{\{p\},(J\cup\{i\})\setminus\{p\}} + \\
& + \frac{\partial f_J}{\partial z_p}(\tau)\overline{\mathcal{U}}_i(\tau,z)\rho_{\{p\},J\setminus\{p\}}\cdot \rho_{\{i\},(J\setminus\{p\})} \Bigg] d\overline{\tau}\wedge d\tau
\end{aligned}
\tag{8.24.3}
$$

for all $i \notin J, \forall\, p \in J$.

8.25 Identity (8.14.4) for a d.f. of a fixed degree

Quite similar reasonings, applied to the identity (8.14.4) with $f = f_J \cdot d\overline{z}^J$ for $|J| = q$ fixed, lead to its following equivalent form:

$$
\begin{aligned}
\int_\Gamma \Bigg\{ & \sum_{\substack{p<j \\ \{p,j\}\subset J}} \Big((-1)^{j-1}\overline{\mathcal{U}}_p(\tau,z) d\overline{\tau}_{[j]}\wedge d\tau - \\
& - (-1)^{p-1}\overline{\mathcal{U}}_j(\tau,z) d\overline{\tau}_{[p]}\wedge d\tau \Big) f_J(\tau)\cdot \rho_{\{p,j\},J}\wedge d\overline{z}^{J\cup\{p,j\}} +
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^m \sum_{\substack{p<j \\ \{p,j\}\subset J}} \Big((-1)^{j-1} \mathcal{U}_p(\tau,z) d\overline{\tau} \wedge d\tau_{[j]} - \\
&\qquad -(-1)^{p-1} \mathcal{U}_j(\tau,z) d\overline{\tau} \wedge d\tau_{[q]} \Big) \cdot \\
&\qquad \cdot f_J(\tau) \cdot \rho_{\{j\},J\setminus\{j\}} \cdot \rho_{\{p\},J\setminus\{p,j\}} \cdot d\overline{z}^{J\setminus\{p,j\}} \Big\} - \\
&- (2i)^m \int_{\Omega^+} \Big\{ \sum_{\substack{p<j \\ \{p,j\}\not\subset J}} \Big(\overline{\mathcal{U}}_p(\tau,z) \cdot \frac{\partial f_J}{\partial \overline{z}_j}(\tau) - -\overline{\mathcal{U}}_j(\tau,z) \frac{\partial f_J}{\partial \overline{z}_p}(\tau) \Big) \\
&\qquad \rho_{\{p,j\},J} \cdot d\overline{z}^{J\cup\{p,j\}} + \\
&+ \sum_{\substack{p<j \\ \{p,j\}\subset J}} \Big(\mathcal{U}_p(\tau,z) \cdot \frac{\partial f_J}{\partial z_j}(\tau) - \mathcal{U}_j(\tau,z) \cdot \frac{\partial f_J}{\partial z_p}(\tau) \Big) \cdot \\
&\qquad \cdot \rho_{\{j\},J\setminus\{j\}} \cdot \rho_{\{p\},J\setminus\{p,j\}} \cdot d\overline{z}^{J\setminus\{p,j\}} \Big\} dV_\tau .
\end{aligned}
\tag{8.25.1}
$$

The above means that (8.14.4) is equivalent to the following set of identities in Ω^+:
if $p<j$ and $\{p,j\}\not\subset J$ then

$$
\begin{aligned}
0 \;=\; & \int_\Gamma \left((-1)^{j-1} \overline{\mathcal{U}}_p(\tau,z)\, d\overline{\tau}_{[j]} \wedge d\tau - \right. \\
& \left. - (-1)^{p-1} \overline{\mathcal{U}}_j(\tau,z)\, d\overline{\tau}_{[p]} \wedge d\tau \right) f_J(\tau) - \\
& - (2i)^m \int_{\Omega^+} \left(\overline{\mathcal{U}}_p(\tau,z) \frac{\partial f_J}{\partial \overline{z}_j}(\tau) - \overline{\mathcal{U}}_j(\tau,z) \frac{\partial f_J}{\partial \overline{z}_p}(\tau) \right) dV_\tau ;
\end{aligned}
\tag{8.25.2}
$$

if $p<j$ and $\{p,j\}\subset J$ then

$$
0 \;=\; (-1)^m \int_\Gamma \left((-1)^{j-1} \mathcal{U}_p(\tau,z)\, d\overline{\tau} \wedge d\tau_{[j]} - \right.
$$

$$- (-1)^{p-1}\mathcal{U}_j(\tau,z)\, d\overline{\tau}\wedge d\tau_{[p]}\big)\, f_J(\tau) -$$

$$-(2i)^m \int\limits_{\Omega^+} \left(\mathcal{U}_p(\tau,z)\,\frac{\partial f_J}{\partial z_j}(\tau) - \mathcal{U}_j(\tau,z)\,\frac{\partial f_J}{\partial z_p}(\tau)\right) dV_\tau. \tag{8.25.3}$$

8.26 About the Koppelman formula

The Koppelman formula has several forms of representation; we shall make use of that one which is in [R]. Let f be an arbitrary $(0,q)$-differential form, with $0\le q\le m$, then in Ω

$$\begin{aligned} f(z) &= \int_\Gamma f(\tau)\wedge K_q(\tau,z) - \int_\Omega \overline{\partial} f(\tau)\wedge K_q(\tau,z) - \\ &\quad - \overline{\partial}_z \int_\Omega f(\tau)\wedge K_{q-1}(\tau,z), \end{aligned} \tag{8.26.1}$$

where $K_{-1}:\equiv 0$,

$$\begin{aligned} K_q(\tau,z) &= \frac{(-1)^{\frac{q(q-1)}{2}}}{(2\pi i)^m}\frac{(m-1)!}{q!(m-q-1)!}\beta^{-n}\cdot \\ &\quad \cdot \partial_\tau\beta\wedge\left(\overline{\partial}_\tau\partial_\tau\beta\right)^{m-q-1}\wedge\left(\overline{\partial}_z\partial_\tau\beta\right)^q, \end{aligned}$$

with $\beta(\tau,z):=|\tau-z|^2$.

Since

$$\begin{aligned} \partial_\tau\beta &= \sum_{p=1}^m(\overline{\tau}_p-\overline{z}_p)d\tau_p; \\ \overline{\partial}_\tau\partial_\tau\beta &= \sum_{p=1}^m d\overline{\tau}_p\wedge d\tau_p; \\ \overline{\partial}_z\partial_\tau\beta &= -\sum_{p=1}^m d\overline{z}_p\wedge d\tau_p; \end{aligned}$$

one gets

$$\left(\overline{\partial}_\tau \partial_\tau \beta\right)^{m-q-1} = (m-q-1)! \sum_{|\mu|=m-q-1} \wedge_{\nu\in\mu}\left(d\overline{\tau}_\nu \wedge d\tau_\nu\right);$$

$$\left(\overline{\partial}_z \partial_\tau \beta\right)^{q} = (-1)^q \cdot q! \sum_{|J|=q} \wedge_{\nu\in J}\left(d\tau_\nu \wedge d\overline{z}_\nu\right).$$

Combining all this, one obtains

$$\begin{aligned}
K_q(\tau,z) &= (-1)^{\frac{q(q+1)}{2}} \sum_{i=1}^{m} \mathcal{U}_i(\tau,z) d\tau_i \wedge \sum_{|I|=m-q-1} d\overline{\tau}_{I_1} \wedge \\
&\quad \wedge d\tau_{I_1} \wedge \ldots \wedge d\overline{\tau}_{I_{m-q-1}} \wedge d\tau_{I_{m-q-1}} \wedge \sum_{|H|=q} d\tau_{H_1} \wedge \\
&\quad \wedge d\overline{z}_{H_1} \wedge \ldots \wedge d\overline{z}_{H_q} \wedge d\tau_{H_q} = \\
&= (-1)^{\frac{q(q+1)}{2}} (-1)^{\frac{(m-q-1)(m-q-2)}{2}} \sum_{i=1}^{m} \mathcal{U}_i(\tau,z) d\tau_i \wedge \\
&\quad \wedge \sum_{|I|=m-q-1} d\overline{\tau}_I \wedge d\tau_I \wedge \sum_{|H|=q} d\tau_H \wedge d\overline{z}_H = \\
&= (-1)^{\frac{q(q+1)}{2}} (-1)^{\frac{(m-q-1)(m-q-2)}{2}} \cdot \sum_{i=1}^{m} \mathcal{U}_i(\tau,z) d\tau_i \wedge \\
&\quad \wedge \sum_{\substack{|\alpha|=m-q-1 \\ |\beta|=q \\ \alpha\cap\beta=\phi}} \rho_{\alpha,\beta} d\overline{\tau}_\alpha \wedge d\tau_{\alpha\cup\beta} \wedge d\overline{z}_\beta = \\
&= (-1)^{\frac{q(q+1)}{2}+\frac{(m-q-1)(m-q-2)}{2}+(m-q-1)} \cdot \sum_{i=1}^{m} \sum_{\substack{|\alpha|=m-q-1 \\ |\beta|=q \\ \alpha\cap\beta\cap\{i\}=\phi}} \\
&\quad \cdot \rho_{\alpha,\beta} (-1)^{i-1} \mathcal{U}_i(\tau,z) d\overline{\tau}_\alpha \wedge d\tau \wedge d\overline{z}_\beta = \\
&= (-1)^{\frac{q(q+1)}{2}+\frac{(m-q-1)(m-q)}{2}} \sum_{i=1}^{m} \sum_{\substack{|\alpha|=m-q-1 \\ |\mu|=q \\ \alpha\cap\beta\cap\{i\}=\phi}} \rho_{\alpha,\beta} (-1)^{i-1} \\
&\quad \cdot \mathcal{U}_i(\tau,z) d\overline{\tau}_\alpha \wedge d\tau \wedge d\overline{z}_\beta =
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\frac{1}{2}m(m-1)+qm} \sum_{i=1}^{m} \sum_{\substack{|\alpha|=m-q-1 \\ |\beta|=q \\ \alpha\cap\beta\cap\{i\}=\phi}} \rho_{\alpha,\beta}(-1)^{i-1} \\
&\quad \cdot \mathcal{U}_i(\tau,z) d\overline{\tau}_\alpha \wedge d\tau \wedge d\overline{z}_\beta .
\end{aligned}
$$

Let again $f = f_J d\overline{\tau}^J$, with $|J| = q$, $q \in \{0, 1, \ldots, m\}$. Hence

$$
\overline{\partial} f = \sum_{k=1}^{m} \frac{\partial f_J}{\partial \overline{\tau_k}} d\overline{\tau}_k \wedge d\overline{\tau}^J = \sum_{k=1}^{m} \frac{\partial f_J}{\partial \overline{\tau}_k} \rho_{\{k\},J} d\overline{\tau}^{J\cup\{k\}}
$$

and

$$
\begin{aligned}
f(\tau) \wedge K_{q-1}(\tau,z) &= f_J(\tau) d\overline{\tau}^J \wedge K_{q-1}(\tau,z) = \\
&= (-1)^{\frac{1}{2}m(m-1)+(q-1)m} f_J(\tau) d\overline{\tau}^J \wedge \\
&\quad \wedge \sum_{i=1}^{m} \sum_{\substack{|\alpha|=m-q \\ |\beta|=q-1 \\ \alpha\cap\beta\cap\{i\}=\phi}} \rho_{\alpha,\beta}(-1)^{i-1} \cdot \\
&\quad \cdot \mathcal{U}_i(\tau,z) d\overline{\tau}_\alpha \wedge d\tau \wedge d\overline{z}_\beta .
\end{aligned}
$$

In the last sum the only terms to survive are those corresponding to $\alpha = \{1, \ldots, m\} \setminus J$, $i \in J$, $\beta = J \setminus \{i\}$. Hence

$$
\begin{aligned}
f(\tau) \wedge K_{q-1}(\tau,z) &= (-1)^{\frac{1}{2}m(m-1)+(q-1)m} \cdot f_J(\tau) \cdot \\
&\quad \cdot \sum_{i\in J} \rho_{\{1,\ldots,m\}\setminus J, J\setminus\{i\}} \cdot (-1)^{i-1} \cdot \rho_{J;\{1,\ldots,m\}\setminus J} \cdot \\
&\quad \cdot \mathcal{U}_i(\tau,z) d\overline{\tau} \wedge d\tau \wedge d\overline{z}^{J\setminus\{i\}} .
\end{aligned}
$$

Using now Theorem 8.15, we obtain

$$
\begin{aligned}
&\overline{\partial}_z \int_\Omega f(\tau) \wedge K_{q-1}(\tau,z) = \\
&\quad = (-1)^{\frac{1}{2}m(m-1)+(q-1)m} \sum_{i\in J} (-1)^{i-1} \rho_{\{1,\ldots m,\}\setminus J, J\setminus\{i\}} \cdot \\
&\qquad \cdot \rho_{J;\{1,\ldots,m\}\setminus J} \cdot \overline{\partial}_z \int_\Omega f_J(\tau) \mathcal{U}_i(\tau,z) d\overline{\tau} \wedge d\tau \wedge d\overline{z}_{J\setminus\{i\}} =
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{\frac{1}{2}m(m-1)+(q-1)m}\sum_{i\in J}(-1)^{i-1}\rho_{\{1,\dots,m\}\setminus J,J\setminus\{i\}}\cdot\rho_{J;\{1,\dots,m\}\setminus J}\cdot\\
&\qquad\sum_{k=1}^{m}\frac{\partial}{\partial\overline{z}_k}\int_\Omega f_J(\tau)\mathcal{U}_i(\tau,z)d\overline{\tau}\wedge d\tau\wedge d\overline{z}_k\wedge d\overline{z}_{J\setminus\{i\}}=\\
&= (-1)^{\frac{1}{2}m(m+1)+(q-1)m}\sum_{i\in J}(-1)^{i-1}\cdot\rho_{\{1,\dots,m\}\setminus J,J\setminus\{i\}}\cdot\rho_{J;\{1,\dots,m\}\setminus J}\cdot\\
&\qquad\cdot\left(\sum_{k=i}+\sum_{\substack{k\notin J\\k\neq i}}+\sum_{k\in J\setminus\{i\}}\right)\cdot\left\{(-1)^{m-i}\int_\Gamma f_J(\tau)\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau_{[i]}\right.\\
&\qquad\qquad\left.+\int_\Omega\frac{\partial f_J}{\partial\tau_i}\overline{\mathcal{U}}_k d\overline{\tau}\wedge d\tau\right\}\wedge d\overline{z}_k\wedge d\overline{z}_{J\setminus\{i\}}=\\
&= (-1)^{\frac{1}{2}m(m-1)+(q-1)m}\cdot\sum_{i\in J}(-1)^{i-1}\rho_{\{1,\dots,m\}\setminus J,J\setminus\{i\}}\cdot\rho_{J,\{1,\dots,m\}\setminus J}\cdot\\
&\quad\cdot\left\{\left[(-1)^{m-i}\int_\Gamma f_J(\tau)\overline{\mathcal{U}}_i(\tau,z)d\overline{\tau}\wedge d\tau_{[i]}\right.\right.\\
&\qquad\left.+\int_\Omega\frac{\partial f_J}{\partial\tau_i}\overline{\mathcal{U}}_i(\tau,z)d\overline{\tau}\wedge d\tau\right]\rho_{i,J\setminus\{i\}}d\overline{z}^J+\\
&\qquad+\sum_{k\notin J}\left[(-1)^{m-i}\int_\Gamma f_J(\tau)\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau_{[i]}+\right.\\
&\qquad\left.\left.+\int_\Omega\frac{\partial f_J}{\partial\tau_i}\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau\right]\wedge\rho_{\{k\},J\setminus\{i\}}d\overline{z}^{(J\setminus\{i\})\cup\{k\}}\right\}.
\end{aligned}
$$

Analogous computation applies to $\int_\Gamma f(\tau)\wedge K_q(\tau,z)$ and $\int_\Omega\overline{\partial}f(\tau)\wedge K_q(\tau,z)$, which gives

$$
\begin{aligned}
&\int_\Gamma f(\tau)\wedge K_q(\tau,z)=\\
&= (-1)^{\frac{1}{2}m(m-1)+qm}\sum_{i\in J}\sum_{p\notin J}(-1)^{i-1}\rho_{(\{1\dots m\}\setminus J)\setminus\{p\},(J\setminus\{i\})\cup\{p\}}\cdot\\
&\quad\cdot\rho_{J,(\{1\dots m\}\setminus J)\setminus\{p\}}\int_\Gamma f_J(\tau)\mathcal{U}_i(\tau,z)d\overline{\tau}_{[p]}\wedge d\tau\wedge d\overline{z}^{(J\cup\{p\})\setminus\{i\}}+\\
&\quad+(-1)^{\frac{1}{2}m(m-1)+qm}\sum_{i\notin J}(-1)^{i-1}\rho_{(\{1\dots m\}\setminus J)\setminus\{i\},J}\cdot\rho_{J,(\{1\dots m\}\setminus J)\setminus\{i\}}\cdot
\end{aligned}
$$

$$\cdot \int_\Gamma f_J(\tau)\mathcal{U}_i(\tau,z)d\overline{\tau}_{[i]} \wedge d\tau \wedge d\overline{z}^J$$

and

$$\int_\Omega \overline{\partial} f(\tau) \wedge K_q(\tau,z) = \sum_{k\notin J} \rho_{\{k\},J}(-1)^{\frac{1}{2}m(m-1)+qm} \cdot$$
$$\cdot (-1)^{k-1}\rho_{\{1\ldots m\}\setminus(J\cup k),J} \int_\Omega \frac{\partial f}{\partial\overline{\tau}_k}\mathcal{U}_k(\tau,z)\cdot\rho_{J\cup\{k\},\{1\ldots m\}\setminus(J\cup\{k\})}\cdot$$
$$\cdot\, d\overline{\tau}\wedge d\tau\wedge d\overline{z}^J + \sum_{k\notin J}\rho_{\{k\},J}(-1)^{\frac{1}{2}m(m-1)+qm}\cdot$$
$$\cdot \sum_{i\in J}(-1)^{i-1}\rho_{\{1\ldots m\}\setminus(J\cup\{k\}),(J\setminus\{i\})\cup\{k\}}$$
$$\int_\Omega \frac{\partial f_J}{\partial\overline{\tau}_k}\mathcal{U}_i(\tau,z)\cdot\rho_{J\cup\{k\},\{1\ldots m\}\setminus(J\cup\{k\})}\cdot$$
$$\cdot\, d\overline{\tau}\wedge d\tau\wedge d\overline{z}^{(J\setminus\{i\})\cup\{k\}}.$$

Substituting all this into (8.26.1) and comparing the differentials in both sides, we conclude that (8.26.1) is equivalent to the following system:

$$f_J(z) =$$
$$= (-1)^{\frac{1}{2}m(m-1)+qm}\sum_{i\notin J}(-1)^{i-1}\rho_{\{1\ldots m\}\setminus J)\setminus\{i\},J}\cdot\rho_{J,(\{1\ldots m\}\setminus J)\setminus\{i\}}\cdot$$
$$\cdot\int_\Gamma f_J(\tau)\mathcal{U}_i(\tau,z)d\overline{\tau}_{[i]}\wedge d\tau$$
$$-\sum_{k\notin J}\rho_{\{k\},J}(-1)^{\frac{1}{2}m(m-1)+qm}\cdot(-1)^{k-1}\rho_{\{1\ldots m\}\setminus(J\cup\{k\}),J}\cdot$$
$$\cdot\int_\Omega\frac{\partial f}{\partial\overline{\tau}_k}\mathcal{U}_k(\tau,z)\cdot\rho_{J\cup\{k\},\{1,\ldots,m\}\setminus(J\cup\{k\})}\cdot d\overline{\tau}\wedge d\tau$$
$$-(-1)^{\frac{1}{2}m(m-1)+(q-1)m}\cdot$$
$$\cdot\sum_{i\in J}(-1)^{i-1}\rho_{\{1,\ldots,m\}\setminus J,J\setminus\{i\}}\cdot\rho_{J,\{1,\ldots,m\}\setminus J}\cdot$$

$$\cdot\Big[(-1)^{m-i}\int_\Gamma f_J(\tau)\overline{\mathcal{U}}_i(\tau,z)d\overline{\tau}\wedge d\tau_{[i]}+$$
$$+\int_\Omega\frac{\partial f_J}{\partial\tau_i}\overline{\mathcal{U}}_i(\tau,z)d\overline{\tau}\wedge d\tau\Big]\rho_{i,J\setminus\{i\}}.$$

(8.26.2)

$$0 = (-1)^{\frac{1}{2}m(m-1)+qm}\cdot$$

$$\cdot\sum_{i\in J}\sum_{p\notin J}(-1)^{i-1}\rho_{(\{1\dots m\}\setminus J)\setminus\{p\},(J\setminus\{i\})\cup\{p\}}\cdot\rho_{J,(\{1\dots m\}\setminus J)\setminus\{p\}}\cdot$$
$$\cdot\int_\Gamma f_J(\tau)\mathcal{U}_i(\tau,z)d\overline{\tau}_{[p]}\wedge d\tau\wedge d\overline{z}^{(J\cup\{p\})\setminus\{i\}}-$$
$$-(-1)^{\frac{1}{2}m(m-1)+qm}\cdot$$
$$\cdot\sum_{k\notin J}\rho_{\{k\},J}\cdot\sum_{i\in J}(-1)^{i-1}\rho_{(\{1\dots m\}\setminus(J\cup\{k\}),(J\setminus\{i\})\cup\{k\}}\cdot$$
$$\cdot\rho_{J\cup\{k\},\{1\dots m\}\setminus(J\cup\{k\})}\cdot$$
$$\cdot\int_\Omega\frac{\partial f_J}{\partial\overline{\tau}_k}\mathcal{U}_i(\tau,z)\cdot d\overline{\tau}\wedge d\tau\wedge d\overline{z}^{(J\setminus\{i\})\cup\{k\}}-$$
$$-(-1)^{\frac{1}{2}m(m-1)+(q-1)m}\cdot\sum_{i\in J}(-1)^{i-1}\rho_{\{1\dots m\}\setminus J,J\setminus\{i\}}\cdot\rho_{J,\{1\dots m\}\setminus J}\cdot$$
$$\sum_{k\notin J}\Big[(-1)^{(m-i)}\int_\Gamma f_J(\tau)\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau_{[i]}l;+$$
$$\int_\Omega\frac{\partial f_J}{\partial\tau_i}\overline{\mathcal{U}}_k(\tau,z)d\overline{\tau}\wedge d\tau\Big]\wedge\rho_{\{k\},J\setminus\{i\}}d\overline{z}^{(J\setminus\{i\})\cup\{k\}}.$$

(8.26.3)

8.27 Auxiliary computations

Consider the right-hand-side term in the representation (8.26.2) for $f_J(z)$. Using the properties of $\rho_{H,K}$ from Subsection 8.23 we have

$$\begin{aligned}(-1)^{\frac{1}{2}m(m-1)+qm}\cdot(-1)^{i-1}\rho_{(\{1...m\}\setminus J)\setminus\{i\},J}\cdot\rho_{J,(\{1...m\}\setminus J)\setminus\{i\}} &= \\ = \quad (-1)^{\frac{1}{2}m(m-1)+qm}\cdot(-1)^{i-1}\cdot(-1)^{q(m-q-1)} &= \\ = \quad (-1)^{\frac{1}{2}m(m-1)}\cdot(-1)^{i-1}.\end{aligned}$$

Analogously, consider

$$\begin{aligned}Y \quad := \quad -(-1)^{\frac{1}{2}m(m-1)+qm}\cdot(-1)^{i-1}\cdot(-1)^{m-i} \\ \cdot\rho_{\{1,...,m\}\setminus J,J\setminus\{i\}}\cdot\rho_{J,\{1,...,m\}\setminus J}\cdot\rho_{\{i\},J\setminus\{i\}},\end{aligned}$$

which, with the intermediate notations $H := \{1,...,m\}\setminus J$, $K := J\setminus\{i\}$, $L := \{i\}$, becomes

$$\begin{aligned}Y \quad = \quad -(-1)^{\frac{1}{2}m(m-1)+(q-1)m}\cdot(-1)^{m-i}\cdot(-1)^{i-1} \\ \cdot\rho_{H,K}\cdot\rho_{K\cup L,H}\cdot\rho_{L,K}.\end{aligned}$$

But $(-1)^{i-1} = \rho_{L,H\cup K} = (-1)^{cardL\cdot card(H\cup K)}\cdot\rho_{H\cup K,L}$ $= (-1)^{m-1}\rho_{H\cup K,L}$, which implies that

$$\begin{aligned}Y \quad = \quad -(-1)^{\frac{1}{2}m(m-1)+(q-1)m}\cdot(-1)^{m-i}\cdot(-1)^{m-1} \\ \cdot\rho_{H\cup K,L}\cdot\rho_{H,K}\cdot\rho_{K\cup L,H}\cdot\rho_{L,K}\end{aligned}$$

and then

$$\begin{aligned}Y \quad = \quad (-1)^{\frac{1}{2}m(m-1)+qm}\cdot(-1)^{m-i}\cdot\rho_{K,L}\cdot \\ \cdot\rho_{H,K\cup L}\cdot\rho_{(K\cup L),H}\cdot\rho_{L,K} = \\ = \quad (-1)^{\frac{1}{2}m(m-1)+qm+m-i}\cdot(\rho_{K,L}\cdot\rho_{L,K})\cdot \\ \cdot(\rho_{H,K\cup L}\cdot\rho_{K\cup L,H}) =\end{aligned}$$

$$= \quad (-1)^{\frac{1}{2}m(m-1)+qm+m-i} \cdot (-1)^{\text{card } K \cdot \text{card } L} \quad \cdot$$

$$\cdot (-1)^{\text{card } H \cdot \text{card}(K \cup L)} =$$

$$= \quad (-1)^{\frac{1}{2}m(m-1)+qm+m-i+q-1+(m-q)\cdot q} =$$

$$= \quad (-1)^{\frac{1}{2}m(m-1)} \cdot (-1)^{m} \cdot (-1)^{i-1}.$$

In the same way we get

$$Y_1 \quad : \quad = -\rho_{\{k\},J} \cdot (-1)^{\frac{1}{2}m(m+1)+qm} \cdot (-1)^{k-1} \quad \cdot$$

$$\cdot \rho_{\{1,\ldots,m\}\setminus(J\cup\{k\}),J} \cdot \rho_{J\cup\{k\},\{1\ldots m\}\setminus(J\cup\{k\})} =$$

$$= \quad -(-1)^{\frac{1}{2}m(m-1)+qm+1} \cdot \rho_{\{k\},J} \cdot \rho_{\{k\},\{1\ldots m\}\setminus\{k\}} \quad \cdot$$

$$\cdot \rho_{\{1\ldots m\}\setminus(J\cup\{k\}),J} \cdot \rho_{J\cup\{k\},\{1\ldots m\}\setminus(J\cup\{k\}),}$$

which, with the intermediate notations $H_1 := \{1\ldots m\}\setminus(J \cup \{k\})$, $K_1 := J;\ L_1 := \{k\}$, becomes

$$Y_1 \quad := \quad (-1)^{\frac{1}{2}m(m+1)+qm+1} \cdot \rho_{L_1,K_1} \cdot \rho_{L_1,H_1\cup K_1} \quad \cdot$$

$$\cdot \rho_{H_1,K_1} \cdot \rho_{K_1\cup L_1,H_1} =$$

$$= \quad (-1)^{\frac{1}{2}m(m+1)+qm+1} \cdot (-1)^{cardL_1 \cdot card(H_1\cup K_1)} \quad .$$

$$\cdot (\rho_{H_1\cup K_1,L_1} \cdot \rho_{H_1,K_1}) \cdot \rho_{L_1,K_1} \cdot \rho_{K_1\cup L_1,H_1} =$$

$$= \quad (-1)^{\frac{1}{2}m(m+1)+qm+1+(m-1)} \cdot \rho_{K_1,L_1} \cdot \rho_{H_1,K_1\cup L_1} \quad \cdot$$

$$\cdot \rho_{L_1,K_1} \cdot \rho_{K_1\cup L_1,H_1} =$$

$$= \quad (-1)^{\frac{1}{2}m(m+1)+qm+m} \cdot (-1)^{q} \cdot (-1)^{(m-q-1)(q+1)}$$

$$= \quad -(-1)^{\frac{1}{2}m(m+1)},$$

and

$$Y_2 \quad := \quad -(-1)^{\frac{1}{2}m(m+1)+(q-1)m} \cdot (-1)^{i-1} \cdot \rho_{\{1\dots m\}\setminus J,J\{i\})} \ \cdot$$

$$\cdot \rho_{J,\{1\dots m\}\setminus J} \cdot \rho_{\{i\},J\setminus\{i\}}) =$$

$$= \quad -(-1)^{\frac{1}{2}m(m+1)+(q-1)m} \cdot \rho_{\{i\},\{1\dots m\}\setminus\{i\}} \cdot \rho_{\{1\dots m\}\setminus J,J\setminus\{i\})} \ \cdot$$

$$\cdot \rho_{J,\{1\dots m\}\setminus J} \cdot \rho_{\{i\},J\setminus\{i\}}),$$

which, after quite similar computation, leads to

$$Y_2 = -(-1)^{\frac{1}{2}m(m+1)}.$$

Hence for the volume integral we have

$$-(-1)^{\frac{1}{2}m(m+1)} \int\limits_{\Omega} \left(\sum_{k\notin J} \frac{\partial f}{\partial \overline{\tau}_k} \mathcal{U}_k\left(\tau, z\right) d\overline{\tau} \wedge d\tau \ + \right.$$

$$\left. + \sum_{k\in J} \frac{\partial f}{\partial \tau_k} \cdot \overline{\mathcal{U}}_k(\tau, z) d\overline{\tau} \wedge d\tau \right),$$

and combining all the above we get that the equality (8.26.2) is equivalent to the following one:

$$f_J(z) \quad = \quad (-1)^{\frac{1}{2}m(m+1)} \left(\sum_{i\notin J} (-1)^{i-1} \int\limits_{\Gamma} f_J\left(\tau\right) \mathcal{U}_i\left(\tau, z\right) d\overline{\tau}_{[i]} \wedge d\tau \ + \right.$$

$$\left. + \ (-1)^m \sum_{i\in J} (-1)^{i+1} \int f_J\left(\tau\right) \overline{\mathcal{U}}_i\left(\tau, z\right) d\overline{\tau} \wedge d\tau_{[i]} \right) \ -$$

$$- \ (-1)^{\frac{1}{2}m(m+1)} \sum_{k\notin J} \int\limits_{\Omega} \frac{\partial f_J}{\partial \overline{\tau}_k}\left(\tau\right) \mathcal{U}_k\left(\tau, z\right) d\overline{\tau} \wedge d\tau \ -$$

$$- (-1)^{\frac{1}{2}m(m+1)} \sum_{i\in J} \int\limits_{\Omega} \frac{\partial f_J}{\partial \tau_i}(\tau)\, \mathcal{U}_i(\tau, z)\, d\overline{\tau} \wedge d\tau \ . \tag{8.27.1}$$

8.28 The Koppelman formula for solutions of the complex Hodge-Dolbeault system

By the linearity reasons all above extends readily onto arbitrary differential forms of a fixed degree $q \in \{0, ..., m\}$: $f(z) = \sum\limits_{\substack{J \\ |J|=q}} f_J(\tau)\, d\overline{z}^J$.

Then we may conclude that for any such a differential form the Koppelman formula (8.26.1) is a particular case of the formula (8.14.3) derived from the hyperholomorphic Borel-Pompeiu formula. This means that if the differential form from the Koppelman formula is, additionally, a solution of the complex Hodge-Dolbeault system, then the Koppelman formula turns into the Bochner-Martinelli-type integral representation from Corollary 8.9.

Theorem *Let $q \in \{0, 1, ...m\}$ be fixed; let $f = \sum\limits_{|J|=q} f_J d\overline{z}^J$ be a differential form of class C^1 satisfying the complex Hodge-Dolbeault system in a domain Ω^+ with usual restrictions. Then the Koppelman formula for f does not contain volume integrals, and it reads*

$$\begin{aligned}
&f(z, d\overline{z}) = \\
&= (-1)^{\frac{m(m+1)}{2}} \cdot \\
&\quad \cdot \int\limits_{\Gamma} \Bigg(\sum_{j=1}^{m} (-1)^{j-1} \mathcal{U}_j(\tau, z)\, d\overline{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\overline{z}^j} \wedge d\overline{z}^j \wedge f(\tau, d\overline{z}) \ + \\
&\quad + (-1)^m \sum_{j=1}^{m} (-1)^{j-1} \overline{\mathcal{U}}_j(\tau, z) d\overline{\tau} \wedge d\tau_{[j]} \wedge d\overline{z}^j \wedge \widehat{d\overline{z}^j} \ \wedge \\
&\quad \wedge f(\tau, \overline{z}) \ + \\
&\quad + \sum_{q\neq j} (-1)^{j-1} \mathcal{U}_q(\tau, z)\, d\overline{\tau}_{[j]} \wedge d\tau \wedge \widehat{d\overline{z}^q} \wedge d\overline{z}^j \wedge f(\tau, d\overline{z}) \ +
\end{aligned}$$

$$+(-1)^m \sum_{q\neq j}(-1)^{j-1}\overline{\mathcal{U}}_q(\tau,z)d\overline{\tau}\wedge d\tau_{[j]}\wedge d\overline{z}^q\wedge\widehat{d\overline{z}^j}\wedge f(\tau,d\overline{z})\Bigg),$$

or equivalently, using the notations introduced earlier,

$$\begin{aligned} f(z) \;=\; & \sum_{|\mathbf{j}|=q}\Bigg(U_{\mathbf{j}}\left[f_{\mathbf{j}}\right](z)\cdot d\overline{z}^{\mathbf{j}} \;+ \\ & +\sum_{n\neq p}\Bigg(U_{np}\left[f_{\mathbf{j}}\right]\widehat{d\overline{z}^n}\wedge d\overline{z}^p\wedge d\overline{z}^{\mathbf{j}} \;+ \\ & +\overline{U}_{np}\left[f_{\mathbf{j}}\right]d\overline{z}^n\wedge\widehat{d\overline{z}^p}\wedge d\overline{z}^{\mathbf{j}}\Bigg)\Bigg). \end{aligned}$$

8.29 Appendix: properties of $\rho_{H,K}$

We give here some complements to what is described in Subsection 8.28. Given $n\in\mathbb{N}$, put $\mathbb{N}_n := \{1,...,n\}$. Then any bijection of $\mathbb{N}_n$ is called a **permutation** of degree n, and let $\mathcal{P}_n$ be the set of all permutations of degree n. It is clear that $card\,\mathcal{P}_n = n!$. The composition $(\sigma,\rho)\in\mathcal{P}_n\times\mathcal{P}_n\longrightarrow\sigma\circ\rho$ makes $\mathcal{P}_n$ a group, and we shall write frequently just $\sigma\rho$ instead of $\sigma\circ\rho$. We denote v the neutral element of $\mathcal{P}_n$. The group $\mathcal{P}_n$ is called the symmetric group of order n.

Let $\{i,j\}\subset\mathbb{N}_n$ with $i\neq j$, the permutation $\tau\in\mathcal{P}_n$ defined by $\tau(i) := j$, $\tau(j) := i$, $\tau(k) = k$ for all $k\in\mathbb{N}_n\backslash\{i,j\}$, is called a **transposition** and is denoted sometimes by (i,j).

It is easily verified that any permutation $\sigma\in\mathcal{P}_n$ can be represented as a product of transpositions. Moreover, any such σ is representable as a product of transpositions with consequent indices, i.e., transpositions of the form $(i,i+1)$ with $i\in\mathbb{N}_{n-1}$. For a fixed permutation the number of transpositions that are necessary for such a representation is always even or always odd, which results in the names an **even permutation** and an **odd permutation**, respectively.

Let $\sigma\in\mathcal{P}_n$, define

$$sgn(\sigma) := \begin{cases} 1, \text{ for an even } \sigma; \\ -1, \text{ for an odd } \sigma; \end{cases}$$

$sgn(\sigma)$ is called the **sign of the permutation**. For any two permutations ρ and σ there holds

$$sgn(\sigma\rho) = sgn(\rho\sigma) = sgn\ \rho \cdot sgn\ \sigma\ .$$

The set of all even permutations in $\mathcal{P}_n$ is a subgroup of $\mathcal{P}_n$ which is called the **alternating group of order** n, and for $n \geq 2$ it has order $\frac{n!}{2}$.

Let $\sigma \in \mathcal{P}_n$ and let $\{i, j\} \subset \mathbb{N}_n$ be such that $i < j$; the pair $\{i, j\}$ is called an **inversion of the permutation** σ if $\sigma(i) > \sigma(j)$; compare with Subsection 8.3. If $I(\sigma)$ is the number of all inversions of σ, then $sgn(\sigma) = (-1)^{I(\sigma)}$, which is a way to find out whether a given permutation is even or odd.

We proceed now to the characteristic $\rho_{H,K}$ introduced in Subsection 8.23.

Take $K = H'$ and

$$\begin{aligned} H &= \{i_1, .., i_r\} \text{ with } i_1 < ... < i_r\ , \\ H' &= \{j_{r+1}, .., j_n\} \text{ with } j_{r+1} < ... < j_n\ ; \end{aligned}$$

consider the following permutation:

$$\sigma = \begin{pmatrix} 1, 2, ..., r, r+1, ..., n \\ i_1, i_2, ..., i_r, j_{r+1}, ..., j_n \end{pmatrix}.$$

All i_α are ordered naturally, hence if $\{p, q\} \subset \mathbb{N}_r$ and $p < q$ then (p, q) is not an inversion of σ. In the same way if $\{p, q\} \subset \mathbb{N}_n \setminus \mathbb{N}_r$ and $p < q$ then (p, q) is not an inversion of σ.

Hence, if $\rho \in \mathbb{N}_r$ and $q \in \mathbb{N}_n \setminus \mathbb{N}_r$ then (p, q) is an inversion of σ if and only if $i_p > j_q$, that is, if and only if (i_p, j_q) is an inversion of the pair (H, H'). Thus

$$sgn\ \sigma = (-1)^{I(\sigma)} = (-1)^{\kappa} = \rho_{H,H'}\ .$$

Let now

$$H \cup K = \{k_1, ..., k_{r+s}\}$$

with $k_1, ..., k_{r+s}$, and $\sigma \in \mathcal{P}_{r+s}$ be a unique permutation such that

$$k_{\sigma(1)} = i_1,\ ...,\ k_{\sigma(r)} = i_r,$$

$$k_{\sigma(r+1)} \quad = \quad j_1, \ ..., \ k_{\sigma(r+s)} = j_s.$$

If $\{p, q\} \subset \mathbb{N}_r$ and $p < q$ then there holds

$$k_{\sigma(1)} = i_p < i_q = k_{\sigma(q)},$$

which means that (p, q) is not an inversion of σ. In the same way, if $\{p, q\} \subset \mathbb{N}_n \setminus \mathbb{N}_r$ and $p < q$, then (p, q) is not an inversion of σ.

Taking now $p \in \mathbb{N}_r$ and $r + q \in \mathbb{N}_s \setminus \mathbb{N}_r$, we may conclude that $(p, r + q)$ will be an inversion of σ if and only if $\sigma(p) > \sigma(r + q)$, which means by definition that $k_{\sigma(p)} > k_{\sigma(r+q)}$, i.e., $i_p > j_q$ which is equivalent to the statement that (i_p, j_q) is an inversion of the pair (H, H').

Let us prove here the first property of $\rho_{H,K}$ from Subsection 8.23. Let H, K, L, be subsets of $\mathbb{N}_n$.

Assume first that H, K, L, are not disjoint in pairs. Then each term in (8.23.3) is equal to zero, and that is all. To prove that the extreme terms in (8.23.3) are equal, it is enough to prove that

$$\rho_{H\cup K,L} = \rho_{H,L} \cdot \rho_{K,L}. \tag{8.29.1}$$

Let κ_1 and κ_2 be the numbers of inversions of the pairs (H, L) and (K, L) . Since $H \cap K = \emptyset$, $\kappa_1 + \kappa_2$ is the number of inversions of the pair $(H \cup K, L)$, hence

$$\rho_{H\cup K,L} = (-1)^{\kappa_1+\kappa_2} = (-1)^{\kappa_1} \cdot (-1)^{\kappa_2} = \rho_{H,L} \cdot \rho_{K,L},$$

and (8.29.1) is true.

In the same way the middle term and the third term in (8.23.1) are equal, which means that (8.23.3) is true also.

Now about the equality (8.23.2). Let here κ_1 and κ_2 be the numbers of inversions of the pairs (H, K) and (K, H) .

Since $H\cap K = \emptyset$, $\kappa_1+\kappa_2$ is the number of all the pairs of elements of $H \times K$, i.e., $\kappa_1 + \kappa_2 = |H| \cdot |K|$. Now

$$\rho_{H,K} \cdot \rho_{K,H} = (-1)^{\kappa_1} \cdot (-1)^{\kappa_2} = (-1)^{\kappa_1+\kappa_2} = (-1)^{card\ H\cdot card\ K}.$$

Chapter 9

Hyperholomorphic theory and Clifford analysis

9.1 One way to introduce a complex Clifford algebra

Recall that the principal discomfort in the above theory consists of the fact that the differentials $d\bar{z}^j$ and the operators $\widehat{d\bar{z}^j}$ are of different natures, and thus, formally, do not belong to the same algebra. To include into the same algebra all complex **d.f.** and the complex algebra generated by the operators $\widehat{d\bar{z}^j}$, so that the equalities (1.3.4) hold, let us consider the following complex algebra generated by i^1, ..., i^m; $\hat{i}^1$, ..., $\hat{i}^m$; with the following rules of multiplication:

$$\left\{ \begin{array}{rcl} i^j \cdot i^k & = & -i^k \cdot i^j, \\ i^j \cdot i^j & = & 0, \end{array} \right. \tag{9.1.1}$$

$$\left\{ \begin{array}{rcl} \hat{i}^j \cdot \hat{i}^k & = & -\hat{i}^k \cdot \hat{i}^j, \\ \hat{i}^j \cdot \hat{i}^j & = & 0, \end{array} \right. \tag{9.1.2}$$

$$\left\{ \begin{array}{rcl} \hat{i}^j \cdot i^k & = & -i^k \cdot \hat{i}^j, \\ \hat{i}^j \cdot i^j + i^j \cdot \hat{i}^j & = & 1, \end{array} \right. \tag{9.1.3}$$

where k, $j = 1, \ldots,$ m; with $k \neq j$.

This complex algebra is associative, distributive, non-commutative, with zero divisors and with identity. We will denote this algebra by $\mathbb{W}_m$.

Immediate consequences of the rules (9.1.1), (9.1.2) and (9.1.3) are that for any $j = 1, \ldots, m$, the following equalities hold:

$$\begin{aligned} \hat{i}^j \cdot i^j \cdot \hat{i}^j \cdot i^j &= \hat{i}^j \cdot i^j, \\ i^j \cdot \hat{i}^j \cdot i^j \cdot \hat{i}^j &= i^j \cdot \hat{i}^j. \end{aligned}$$

Each element $\mathbf{a}$ of $\mathbb{W}_m$ is of the form

$$\mathbf{a} = \sum_{\mathbf{jk}} \mathbf{a}_{\mathbf{jk}} \hat{i}^{\mathbf{j}} \cdot i^{\mathbf{k}}, \tag{9.1.4}$$

where $\mathbf{a}_{\mathbf{jk}} \in \mathbb{C}$ and $\mathbf{j}$, $\mathbf{k}$ are, respectively, strictly increasing $|\mathbf{j}|$-tuple, $|\mathbf{k}|$-tuple, in $\{1, \ldots, m\}$ with $|\mathbf{j}|, |\mathbf{k}| = 0, \ldots, m$, and

$$\begin{aligned} \hat{i}^{\mathbf{j}} &:= \hat{i}^{j_1} \cdot \ldots \cdot \hat{i}^{j_{|\mathbf{j}|}}, \\ i^{\mathbf{k}} &:= i^{k_1} \cdot \ldots \cdot i^{k_{|\mathbf{k}|}}. \end{aligned}$$

Note that $\mathbf{j}$ or $\mathbf{k}$ can be equal $\emptyset$, the empty set; in this case, $\hat{i}^{\mathbf{j}}$ or $i^{\mathbf{k}}$ are not included in the expression $\hat{i}^{\mathbf{j}} \cdot i^{\mathbf{k}}$.

The defining equalities (9.1.1), (9.1.2) and (9.1.3) say that two Grassmann algebras, one generated by $\{i^k\}$ and the other by $\left\{\hat{i}^k\right\}$, got mixed by the crucial conditions (9.1.3) into a single object.

9.1.1 Classical definition of a complex Clifford algebra

Let $Cl_{0,\,2m}$ be a complex Clifford algebra with generators e_1, e_2, ..., e_{2m}. This means that

$$\begin{aligned} e_k^2 = -1 =: -e_0, &\qquad k \in \{1, \ldots, 2m\}, \\ e_k e_q + e_q e_k = 0, &\qquad k \neq q, \end{aligned}$$

any element $a \in Cl_{0,\,2m}$ is of the form

$$a = \sum_A a_A e_A,$$

where $A = (\alpha_1, \ldots, \alpha_p)$ with $1 \leq \alpha_1 < \ldots < \alpha_p \leq 2m$, $\{a_A\} \subset \mathbb{C}$, $e_A := e_{\alpha_1} \ldots e_{\alpha_p}$. The reader can find all the necessary information in [DeSoSo] and in many other sources.

Mention that for the Clifford conjugation of a we use the notation a^*:

$$a^* := \sum_A a_A e_A^*$$

with $e_A^* := e_{\alpha_p}^* \ldots e_{\alpha_1}^* := (-e_{\alpha_p}) \ldots (-e_{\alpha_1})$, and for the complex conjugation $\bar{a}$:

$$\bar{a} := \sum_A \bar{a}_A \cdot e_A$$

with $\bar{a}_A := Re\, a_A - iIm\, a_A$. Note that sometimes both conjugations are denoted by the same symbol, but we prefer to use different ones.

Consider the following elements of $Cl_{0,\,2m}$:

$$\begin{aligned} f_j &:= \frac{1}{2}\left(e_{2j} + ie_{2j-1}\right), \\ \bar{f}_j &= \frac{1}{2}\left(e_{2j} - ie_{2j-1}\right), \end{aligned}$$

where $j = 1, \ldots, m$. Note that the $2m$-tuple $\left(f_1, \ldots, f_m, -\bar{f}_1, \ldots, -\bar{f}_m\right)$ is an underlying space basis of $Cl_{0,\,2m}$ in the sense of a complex vector space. But, on the other hand,

$$\left.\begin{aligned} f_j \cdot f_k &= -f_k \cdot f_j, \\ \left(-\bar{f}_j\right) \cdot f_k &= -f_k \cdot \left(-\bar{f}_j\right), \\ \left(-\bar{f}_j\right) \cdot \left(-\bar{f}_k\right) &= -\left(-\bar{f}_k\right) \cdot \left(-\bar{f}_j\right), \end{aligned}\right\} \quad \begin{array}{l}\text{for all } k, j = 1, \\ \quad \ldots, m; \\ \text{with } k \neq j;\end{array}$$

$$\left.\begin{aligned} f_j \cdot f_j &= 0, \\ \left(-\bar{f}_j\right) \cdot f_j + f_j \cdot \left(-\bar{f}_j\right) &= 1, \\ \left(-\bar{f}_j\right) \cdot \left(-\bar{f}_j\right) &= 0, \end{aligned}\right\} \quad \text{for all } j = 1, \ldots, m. \tag{9.1.5}$$

Hence we can make the following identifications: $i^j \equiv f_j$ and $\hat{i}^j \equiv -\bar{f}_j$.

This means that the complex algebra $\mathbb{W}_m$ is nothing more than the complex Clifford algebra $Cl_{0,\,2m}$, but with the other basis fixed.

This basis is called the Witt basis and below we will see that it is important to study the Grassmann algebra as a part of the Clifford algebra. We shall use the notation $\mathbb{W}_m$ when we want to use the representation (9.1.4) of a complex Clifford number.

9.2 Some differential operators on $\mathbb{W}_m$-valued functions

Let Ω be an open set on $\mathbb{C}^m$. We shall consider $\mathbb{W}_m$-valued functions defined in Ω:

$$\mathbf{f} : \Omega \to \mathbb{W}_m .$$

On the set $C^1(\Omega;\ \mathbb{W}_m)$ define the operators $\bar{\boldsymbol{\partial}}$ and $\bar{\boldsymbol{\partial}}^{\wedge}$ by the equalities

$$\bar{\boldsymbol{\partial}}[\mathbf{f}] := \sum_{q=1}^{m} i^q \frac{\partial \mathbf{f}}{\partial \bar{z}_q}, \tag{9.2.1}$$

$$\bar{\boldsymbol{\partial}}^{\wedge}[\mathbf{f}] := \sum_{q=1}^{m} \hat{i}^q \frac{\partial \mathbf{f}}{\partial z_q}, \tag{9.2.2}$$

where

$$\frac{\partial}{\partial \bar{z}_q} := \frac{1}{2}\left(\frac{\partial}{\partial x_q} + i\frac{\partial}{\partial y_q}\right), \qquad \frac{\partial}{\partial z_q} := \frac{1}{2}\left(\frac{\partial}{\partial x_q} - i\frac{\partial}{\partial y_q}\right).$$

As in Subsection 1.3, $\mathbb{W}_m$-valued function $\bar{\boldsymbol{\partial}}[\mathbf{f}]$ can be interpreted as a specific "Clifford product" of a $\mathbb{W}_m$-valued function $\mathbf{f}$ with the "$\mathbb{W}_m$-valued function whose coordinates are partial derivations (not partial derivatives)", i.e., with $\bar{\boldsymbol{\partial}} := \sum\limits_{q=1}^{m} i^q \frac{\partial}{\partial \bar{z}_q}$; what is more, in this sense $\mathbf{f}$ is multiplied by $\bar{\boldsymbol{\partial}}$ on the left-hand side:

$$\bar{\boldsymbol{\partial}} \cdot \mathbf{f} := \bar{\boldsymbol{\partial}}[\mathbf{f}]. \tag{9.2.3}$$

Of course it is assumed here that a scalar-valued function commutes with the generators of $\mathbb{W}_m$. The same interpretations are valid for $\bar{\boldsymbol{\partial}}^{\wedge}$.

We define now

$$\begin{aligned}\bar{\boldsymbol{\partial}}_r\,[\mathbf{f}] &:= \mathbf{f}\cdot\bar{\boldsymbol{\partial}} := \sum_{q=1}^{m}\frac{\partial \mathbf{f}}{\partial \bar{z}_q} i^q,\\ \bar{\boldsymbol{\partial}}_r^{\wedge}\,[\mathbf{f}] &:= \mathbf{f}\cdot\bar{\boldsymbol{\partial}}^{\wedge} := \sum_{q=1}^{m}\frac{\partial \mathbf{f}}{\partial z_q}\hat{i}^q.\end{aligned}$$

Note that $\bar{\boldsymbol{\partial}}_r\,[\mathbf{f}]$ and $\bar{\boldsymbol{\partial}}_r^{\wedge}\,[\mathbf{f}]$ are $\mathbb{W}_m$-valued functions, which differs greatly from the definition of $\bar{\partial}_r^*$ in Subsection 1.3. Mention that now one can use the notation $\mathbf{f}\cdot\bar{\boldsymbol{\partial}}^{\wedge}$ without possible confusions (compare with Subsection 1.3), but one needs to be careful with the fact that the multiplication "$\cdot$", when the factors $\bar{\boldsymbol{\partial}}$ and $\bar{\boldsymbol{\partial}}^{\wedge}$ are included, is not associative, that is, in general:

$$\begin{aligned}\left(\mathbf{f}\cdot\bar{\boldsymbol{\partial}}\right)\cdot\mathbf{g} &\neq \mathbf{f}\cdot\left(\bar{\boldsymbol{\partial}}\cdot\mathbf{g}\right),\\ \left(\mathbf{f}\cdot\bar{\boldsymbol{\partial}}^{\wedge}\right)\cdot\mathbf{g} &\neq \mathbf{f}\cdot\left(\bar{\boldsymbol{\partial}}^{\wedge}\cdot\mathbf{g}\right).\end{aligned}$$

9.2.1 Factorization of the Laplace operator

On $\mathbb{W}_m$-valued functions of class C^2 it is natural to define the complex Laplace operator as

$$\Delta_{\mathbb{C}^m}\,[\mathbf{f}] := \sum_{k=1}^{m}\frac{\partial^2\mathbf{f}}{\partial z_k\partial\bar{z}_k} = \sum_{k=1}^{m}\frac{\partial^2\mathbf{f}}{\partial\bar{z}_k\partial z_k} \tag{9.2.4}$$

(compare with (1.3.3)).

Theorem *The following operator equalities hold on $\mathbb{W}_m$-valued functions of class C^2:*

$$\begin{aligned}\bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge} + \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}} &= \Delta_{\mathbb{C}^m},\\ \bar{\boldsymbol{\partial}}_r\bar{\boldsymbol{\partial}}_r^{\wedge} + \bar{\boldsymbol{\partial}}_r^{\wedge}\bar{\boldsymbol{\partial}}_r &= \Delta_{\mathbb{C}^m}.\end{aligned} \tag{9.2.5}$$

$$\begin{aligned}\bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}} &= 0,\\ \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}^{\wedge} &= 0,\\ \bar{\boldsymbol{\partial}}_r\bar{\boldsymbol{\partial}}_r &= 0,\\ \bar{\boldsymbol{\partial}}_r^{\wedge}\bar{\boldsymbol{\partial}}_r^{\wedge} &= 0.\end{aligned} \tag{9.2.6}$$

Proof. We have

$$
\begin{aligned}
\left(\bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge}+\bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}\right)[\mathbf{f}] &:= \sum_{p=1}^{m} i^{p}\frac{\partial}{\partial\bar{z}_{p}}\bar{\partial}^{*}[\mathbf{f}]+\sum_{p=1}^{m}\hat{i}^{p}\frac{\partial}{\partial z_{p}}\bar{\partial}[\mathbf{f}]= \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} i^{p}\hat{i}^{q}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}+\sum_{p=1}^{m}\sum_{q=1}^{m}\hat{i}^{p}i^{q}\frac{\partial^{2}\mathbf{f}}{\partial z_{p}\partial\bar{z}_{q}}= \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} i^{p}\hat{i}^{q}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}+\sum_{q=1}^{m}\sum_{p=1}^{m}\hat{i}^{q}i^{p}\frac{\partial^{2}\mathbf{f}}{\partial z_{q}\partial\bar{z}_{p}}= \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} i^{p}\hat{i}^{q}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}+\sum_{q=1}^{m}\sum_{p=1}^{m}\hat{i}^{q}i^{p}\frac{\partial^{2}\mathbf{f}}{\partial z_{q}\partial\bar{z}_{p}}.
\end{aligned}
$$

Recall that $\mathbf{f}$ is a function of class C^{2}, then

$$
\begin{aligned}
\left(\bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge}+\bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}\right)[\mathbf{f}] &= \sum_{p=1}^{m}\sum_{q=1}^{m} i^{p}\hat{i}^{q}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}+\sum_{p=1}^{m}\sum_{q=1}^{m}\hat{i}^{q}i^{p}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}= \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m}\left\{i^{p}\hat{i}^{q}+\hat{i}^{q}i^{p}\right\}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}= \\
&= \sum_{p=1}^{m}\left\{i^{p}\hat{i}^{p}+\hat{i}^{p}i^{p}\right\}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{p}}+ \\
&+ \sum_{\substack{p,\ q=1,\ \ldots,\ m\\ p<q}}\left\{i^{p}\hat{i}^{q}+\hat{i}^{q}i^{p}\right\}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}.
\end{aligned}
$$

Now using the equalities (9.1.1), (9.1.2) and (9.1.3), we get

$$
\begin{aligned}
\left(\bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge}+\bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}\right)[\mathbf{f}] &= \sum_{p=1}^{m}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{p}}+\sum_{\substack{p,q=1,\ldots,m\\ p<q}}\left\{i^{p}\hat{i}^{q}-i^{p}\hat{i}^{q}\right\}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial z_{q}}= \\
&=: \Delta_{\mathbb{C}^{m}}[\mathbf{f}].
\end{aligned}
$$

The same for the second equality in (9.2.5).

We have now

$$
\bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}[\mathbf{f}] := \sum_{p=1}^{m} i^{p}\frac{\partial}{\partial\bar{z}_{p}}\bar{\partial}[\mathbf{f}]=\sum_{p=1}^{m}\sum_{q=1}^{m} i^{p}i^{q}\frac{\partial^{2}\mathbf{f}}{\partial\bar{z}_{p}\partial\bar{z}_{q}}=
$$

$$= \sum_{p=1}^{m} i^p i^p \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_p} +$$

$$+ \sum_{\substack{p,\ q=1,\ \ldots,\ m \\ p<q}} \left\{ i^p i^q \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_q} + i^q i^p \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_q \partial \bar{z}_p} \right\}.$$

Recall that $\mathbf{f}$ is a function of class C^2, then

$$\bar{\boldsymbol{\partial}} \bar{\boldsymbol{\partial}} [\mathbf{f}] := \sum_{p=1}^{m} i^p i^p \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_p} +$$

$$+ \sum_{\substack{p,\ q=1,\ \ldots,\ m \\ p<q}} \left\{ i^p i^q \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_q} + i^q i^p \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_q} \right\} =$$

$$= \sum_{p=1}^{m} i^p i^p \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_p} + \sum_{\substack{p,\ q=1,\ \ldots,\ m \\ p<q}} \left\{ i^p i^q + i^q i^p \right\} \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_q}.$$

Now using the equalities (9.1.1), (9.1.2) and (9.1.3), we get

$$\bar{\boldsymbol{\partial}} \bar{\boldsymbol{\partial}} [\mathbf{f}] = \sum_{\substack{p,\ q=1,\ \ldots,\ m \\ p<q}} \left\{ i^p i^q - i^p i^q \right\} \frac{\partial^2 \mathbf{f}}{\partial \bar{z}_p \partial \bar{z}_q} = 0.$$

The same for the rest of equalities in (9.2.6). ■

Note that we don't have the second equality from (9.2.5) in (1.3.4). Note also that the equalities (9.2.5) are not factorizations of the Laplace operator. . There are various ways to obtain them for instance introducing the algebra as below in Subsection 9.4.

9.3 Relation of the operators $\bar{\partial}$ and $\bar{\partial}^{\wedge}$ with the Dirac operator of Clifford analysis

Let Ω be a domain in $\mathbb{C}^m$. For all $f \in C^1(\Omega;\ Cl_{0,\,2m})$ one introduces the Clifford-Dirac operator as follows:

$$\vartheta_t [f] := \sum_{k=1}^{2m} e_k \cdot \frac{\partial f}{\partial t_k}.$$

Note that the usual notation of the Clifford-Dirac operator is ∂_t, but we here prefer to use the notation ϑ_t because the notation ∂ has been used, as a traditional notation, in this text and in several complex variables function theory in many other meanings in fact.

We have

$$\begin{aligned}
\vartheta_t &:= \sum_{k=1}^{2m} e_k \cdot \frac{\partial}{\partial t_k} = \\
&= \sum_{k=1}^{m} e_{2k-1} \cdot \frac{\partial}{\partial x_k} + \sum_{k=1}^{m} e_{2k} \cdot \frac{\partial}{\partial y_k} = \\
&= \sum_{k=1}^{m} -i\left(i^k + \hat{i}^k\right) \cdot \frac{\partial}{\partial x_k} + \sum_{k=1}^{m} \left(i^k - \hat{i}^k\right) \cdot \frac{\partial}{\partial y_k} = \\
&= -i \sum_{k=1}^{m} i^k \left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right) - i \sum_{k=1}^{m} \hat{i}^k \left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) = \\
&= -2i \left(\sum_{k=1}^{m} i^k \frac{\partial}{\partial \bar{z}_k} + \sum_{k=1}^{m} \hat{i}^k \frac{\partial}{\partial z_k}\right) = -2i \left(\bar{\boldsymbol{\partial}} + \bar{\boldsymbol{\partial}}^{\wedge}\right).
\end{aligned}$$

9.4 Matrix algebra with entries from $\mathbb{W}_m$

Now we need the set of 2×2 matrices with entries from $\mathbb{W}_m$. We use the following denotations:

$$\mathfrak{W}_m := \begin{pmatrix} \mathbb{W}_m & \mathbb{W}_m \\ \mathbb{W}_m & \mathbb{W}_m \end{pmatrix} := \left\{ \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{pmatrix} \mid \left\{\mathbf{A}^{ij}\right\} \subset \mathbb{W}_m \right\}.$$

The structure of a complex linear space in $\mathbb{W}_m$ is inherited by $\mathfrak{W}_m$: it is enough to add the elements and to multiply them by complex scalars in an entry-wise manner.

Given $\mathbf{A}$, $\mathbf{B}$ from $\mathfrak{W}_m$, their "Clifford product" $\mathbf{A} \underset{\star}{\cdot} \mathbf{B}$ is introduced as follows:

$$\begin{aligned}
\mathbf{A} \underset{\star}{\cdot} \mathbf{B} &= \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{pmatrix} \underset{\star}{\cdot} \begin{pmatrix} \mathbf{B}^{11} & \mathbf{B}^{12} \\ \mathbf{B}^{21} & \mathbf{B}^{22} \end{pmatrix} \\
&:= \begin{pmatrix} \mathbf{A}^{11} \cdot \mathbf{B}^{11} + \mathbf{A}^{12} \cdot \mathbf{B}^{21}; & \mathbf{A}^{11} \cdot \mathbf{B}^{12} + \mathbf{A}^{12} \cdot \mathbf{B}^{22} \\ \mathbf{A}^{21} \cdot \mathbf{B}^{11} + \mathbf{A}^{22} \cdot \mathbf{B}^{21}; & \mathbf{A}^{21} \cdot \mathbf{B}^{12} + \mathbf{A}^{22} \cdot \mathbf{B}^{22} \end{pmatrix}.
\end{aligned}$$

This product remains to be associative and distributive:

$$\begin{aligned}\left(\mathbf{A} \underset{\star}{\cdot} \mathbf{B}\right) \underset{\star}{\cdot} \mathbf{C} &= \mathbf{A} \underset{\star}{\cdot} \left(\mathbf{B} \underset{\star}{\cdot} \mathbf{C}\right);\\ (\mathbf{A}+\mathbf{B}) \underset{\star}{\cdot} \mathbf{C} &= \mathbf{A} \underset{\star}{\cdot} \mathbf{C} + \mathbf{B} \underset{\star}{\cdot} \mathbf{C},\\ \mathbf{C} \underset{\star}{\cdot} (\mathbf{A}+\mathbf{B}) &= \mathbf{C} \underset{\star}{\cdot} \mathbf{A} + \mathbf{C} \underset{\star}{\cdot} \mathbf{B}.\end{aligned}$$

Thus we shall consider $\mathfrak{W}_m$ as a complex algebra which is associative, distributive, non-commutative, with zero divisors and with identity.

9.5 The matrix Dirac operators

Let Ω be a open set on $\mathbb{C}^m$. We shall consider $\mathfrak{W}_m$-valued functions defined in Ω:

$$\mathbf{F} : \Omega \to \mathfrak{W}_m.$$

Now we need certain matrix operators composed from *scalar* operators (9.2.1), (9.2.2), and acting on $\mathfrak{W}_m$-valued functions of class C^1. We put

$$\overline{\mathbf{D}} := \begin{pmatrix} \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \\ \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \end{pmatrix}, \qquad \overline{\mathbf{D}}^{*} := \begin{pmatrix} \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \\ \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \end{pmatrix}; \tag{9.5.1}$$

i.e., for $\mathbf{F} \in C^1(\Omega;\ \mathbb{W}_m)$ we define $\overline{\mathbf{D}}[\mathbf{F}]$ and $\overline{\mathbf{D}}^{*}[\mathbf{F}]$ to be

$$\begin{aligned}\overline{\mathbf{D}}[\mathbf{F}] &= \begin{pmatrix} \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{11} + \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{21}; & \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{12} + \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{22} \\ \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{11} + \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{21}; & \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{12} + \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{22} \end{pmatrix}\\ &= \begin{pmatrix} \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{11}\right] + \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{21}\right]; & \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{12}\right] + \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{22}\right] \\ \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{11}\right] + \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{21}\right]; & \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{12}\right] + \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{22}\right] \end{pmatrix},\end{aligned} \tag{9.5.2}$$

$$\begin{aligned}\overline{\mathbf{D}}^{*}[\mathbf{F}] &= \begin{pmatrix} \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{11} + \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{21}; & \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{12} + \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{22} \\ \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{11} + \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{21}; & \bar{\boldsymbol{\partial}} \cdot \mathbf{F}^{12} + \bar{\boldsymbol{\partial}}^{\wedge} \cdot \mathbf{F}^{22} \end{pmatrix} =\\ &= \begin{pmatrix} \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{11}\right] + \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{21}\right]; & \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{12}\right] + \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{22}\right] \\ \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{11}\right] + \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{21}\right]; & \bar{\boldsymbol{\partial}}\left[\mathbf{F}^{12}\right] + \bar{\boldsymbol{\partial}}^{\wedge}\left[\mathbf{F}^{22}\right] \end{pmatrix}.\end{aligned} \tag{9.5.3}$$

Recalling observation in Subsection 9.2, we can interpret the matrix $\overline{\mathbf{D}}\,[\mathbf{F}]$ as a result of the *matrix Clifford multiplication* of $\mathbf{F}$ by $\overline{\mathbf{D}}$ on the left-hand side:

$$\overline{\mathbf{D}} \underset{\star}{\cdot} \mathbf{F} := \begin{pmatrix} \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \\ \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \end{pmatrix} \underset{\star}{\cdot} \begin{pmatrix} \mathbf{F}^{11} & \mathbf{F}^{12} \\ \mathbf{F}^{21} & \mathbf{F}^{22} \end{pmatrix}.$$

Now we introduce *the right-hand-side* operator $\overline{\mathbf{D}}_r$ by the rule

$$\begin{aligned} \overline{\mathbf{D}}_r\,[\mathbf{F}] \;&:=\; \begin{pmatrix} \mathbf{F}^{11}\cdot\bar{\boldsymbol{\partial}} + \mathbf{F}^{12}\cdot\bar{\boldsymbol{\partial}}^{\wedge}; & \mathbf{F}^{11}\cdot\bar{\boldsymbol{\partial}}^{\wedge} + \mathbf{F}^{12}\cdot\bar{\boldsymbol{\partial}} \\ \mathbf{F}^{21}\cdot\bar{\boldsymbol{\partial}} + \mathbf{F}^{22}\cdot\bar{\boldsymbol{\partial}}^{\wedge}; & \mathbf{F}^{21}\cdot\bar{\boldsymbol{\partial}}^{\wedge} + \mathbf{F}^{22}\cdot\bar{\boldsymbol{\partial}} \end{pmatrix} = \\ &=\; \begin{pmatrix} \bar{\boldsymbol{\partial}}_r\left[\mathbf{F}^{11}\right] + \bar{\boldsymbol{\partial}}_r^{\wedge}\left[\mathbf{F}^{12}\right]; & \bar{\boldsymbol{\partial}}_r^{\wedge}\left[\mathbf{F}^{11}\right] + \bar{\boldsymbol{\partial}}_r\left[\mathbf{F}^{12}\right] \\ \bar{\boldsymbol{\partial}}_r\left[\mathbf{F}^{21}\right] + \bar{\boldsymbol{\partial}}_r^{\wedge}\left[\mathbf{F}^{22}\right]; & \bar{\boldsymbol{\partial}}_r^{\wedge}\left[\mathbf{F}^{21}\right] + \bar{\boldsymbol{\partial}}_r\left[\mathbf{F}^{22}\right] \end{pmatrix}. \end{aligned} \tag{9.5.4}$$

Matrix (9.5.4) may differ from $\overline{\mathbf{D}} \underset{\star}{\cdot} \mathbf{F} = \overline{\mathbf{D}}\,[\mathbf{F}]$, but in this occasion both are $\mathfrak{W}_m$-valued functions (compare with (2.3.7)). Analogous definitions and conclusions are true for the *right-hand-side* operator $\overline{\mathbf{D}}_r^*$.

For all above, $\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}^*$, as well as their right-hand side counterparts, are also called the matrix Dirac operators.

9.5.1 Factorization of the Laplace operator on $\mathfrak{W}_m$-valued functions

Theorem *The following operator equalities hold on* $C^2\left(\Omega;\ \mathfrak{W}_m\right)$*:*

$$\overline{\mathbf{D}} \circ \overline{\mathbf{D}}^* = \overline{\mathbf{D}}^* \circ \overline{\mathbf{D}} = \Delta_{\mathbb{C}^m}\mathbb{E}_{2\times 2}, \tag{9.5.5}$$

$$\overline{\mathbf{D}}_r \circ \overline{\mathbf{D}}_r^* = \overline{\mathbf{D}}_r^* \circ \overline{\mathbf{D}}_r = \Delta_{\mathbb{C}^m}\mathbb{E}_{2\times 2}. \tag{9.5.6}$$

Proof. We have

$$\begin{aligned} \overline{\mathbf{D}} \circ \overline{\mathbf{D}}^* \;&:=\; \begin{pmatrix} \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \\ \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \end{pmatrix} \underset{\star}{\cdot} \begin{pmatrix} \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \\ \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \end{pmatrix} = \\ &=\; \begin{pmatrix} \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge} + \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}; & \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}} + \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}^{\wedge} \\ \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}^{\wedge} + \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}; & \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}} + \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge} \end{pmatrix}. \end{aligned}$$

Now using Theorem 9.2.1, we get

$$\overline{\mathbf{D}} \circ \overline{\mathbf{D}}^{*} := \begin{pmatrix} \Delta_{\mathbb{C}^m} & 0 \\ 0 & \Delta_{\mathbb{C}^m} \end{pmatrix} = \Delta \mathbb{E}_{2\times 2}.$$

Also, we have

$$\begin{aligned} \overline{\mathbf{D}}_r \circ \overline{\mathbf{D}}_r^{*} &:= \begin{pmatrix} \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \\ \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \end{pmatrix} \underset{\star}{\cdot} \begin{pmatrix} \bar{\boldsymbol{\partial}}^{\wedge} & \bar{\boldsymbol{\partial}} \\ \bar{\boldsymbol{\partial}} & \bar{\boldsymbol{\partial}}^{\wedge} \end{pmatrix} = \\ &= \begin{pmatrix} \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge} + \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}; & \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}} + \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}^{\wedge} \\ \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}}^{\wedge} + \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}; & \bar{\boldsymbol{\partial}}^{\wedge}\bar{\boldsymbol{\partial}} + \bar{\boldsymbol{\partial}}\bar{\boldsymbol{\partial}}^{\wedge} \end{pmatrix}. \end{aligned}$$

Now using Theorem 9.2.1, we get

$$\overline{\mathbf{D}}_r \circ \overline{\mathbf{D}}_r^{*} := \begin{pmatrix} \Delta_{\mathbb{C}^m} & 0 \\ 0 & \Delta_{\mathbb{C}^m} \end{pmatrix} = \Delta \mathbb{E}_{2\times 2}.$$

The rest of the proof is similar to the above. ■

Note that the proofs of Theorem 9.5.1 show that the equalities (9.1.1), (9.1.2) and (9.1.3) are sufficient conditions to obtain the above authentic factorization of the Laplace operator.

9.6 The fundamental solution of the matrix Dirac operators

Let $\theta_{\mathbb{C}^m}$ be a fundamental solution of the complex Laplace operator on $\mathbb{C}^m$, see Subsection 1.3, that is

$$\theta_{\mathbb{C}^m}(z) := \begin{cases} -\frac{4(m-2)!}{(2\pi)^m} \frac{1}{|z|^{2m-2}}, & m > 1, \\ \frac{2}{\pi} ln\,|z|, & m = 1. \end{cases}$$

Then it follows easily that

$$\Delta_{\mathbb{C}^m} \mathbb{E}_{2\times 2} \left[\theta_{\mathbb{C}^m} E_{2\times 2}\right] = \delta_{\mathbb{C}^m} E_{2\times 2},$$

i.e., the matrix

$$\begin{pmatrix} \theta_{\mathbb{C}^m} & 0 \\ 0 & \theta_{\mathbb{C}^m} \end{pmatrix}$$

is a fundamental solution to the Laplace operator $\Delta_{\mathbb{C}^m}\mathbb{E}_{2\times 2}$ on $\mathfrak{W}_m$.

Hence, we have the fundamental solutions of the operators $\overline{\mathbf{D}}$ and $\overline{\mathbf{D}}^*$:

$$\begin{aligned}
\mathbf{K}_{\overline{\mathbf{D}}}(\zeta) &:= \overline{\mathbf{D}}^*\left[\theta_{\mathbb{C}^m}E_{2\times 2}\right](\zeta) = \\
&= \sum_{q=1}^{m}\begin{pmatrix} \frac{\partial\theta_{\mathbb{C}^m}}{\partial\zeta_q}(\zeta)\,\hat{i}^q; & \frac{\partial\theta_{\mathbb{C}^m}}{\partial\bar{\zeta}_q}(\zeta)\,i^q \\ \\ \frac{\partial\theta_{\mathbb{C}^m}}{\partial\bar{\zeta}_q}(\zeta)\,i^q; & \frac{\partial\theta_{\mathbb{C}^m}}{\partial\zeta_q}(\zeta)\,\hat{i}^q \end{pmatrix} = \\
&= 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\begin{pmatrix} \frac{\bar{\zeta}_q}{|\zeta|^{2m}}\hat{i}^q; & \frac{\zeta_q}{|\zeta|^{2m}}i^q \\ \\ \frac{\zeta_q}{|\zeta|^{2m}}i^q; & \frac{\bar{\zeta}_q}{|\zeta|^{2m}}\hat{i}^q \end{pmatrix}, \qquad (9.6.1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_{\overline{\mathbf{D}}^*}(\zeta) &:= \overline{\mathbf{D}}\left[\theta_{\mathbb{C}^m}E_{2\times 2}\right](\zeta) := \\
&:= \sum_{q=1}^{m}\begin{pmatrix} \frac{\partial\theta_{\mathbb{C}^m}}{\partial\bar{\zeta}_q}(\zeta)\,i^q; & \frac{\partial\theta_{\mathbb{C}^m}}{\partial\zeta_q}(\zeta)\,\hat{i}^q \\ \\ \frac{\partial\theta_{\mathbb{C}^m}}{\partial\zeta_q}(\zeta)\,\hat{i}^q; & \frac{\partial\theta_{\mathbb{C}^m}}{\partial\bar{\zeta}_q}(\zeta)\,i^q \end{pmatrix} = \\
&= 2\frac{(m-1)!}{\pi^m}\sum_{q=1}^{m}\begin{pmatrix} \frac{\zeta_q}{|\zeta|^{2m}}i^q; & \frac{\bar{\zeta}_q}{|\zeta|^{2m}}\hat{i}^q \\ \\ \frac{\bar{\zeta}_q}{|\zeta|^{2m}}\hat{i}^q; & \frac{\zeta_q}{|\zeta|^{2m}}i^q \end{pmatrix}. \qquad (9.6.2)
\end{aligned}$$

Formally, for (9.5.6), one can set

$$\begin{aligned}
\mathbf{K}_{\overline{\mathbf{D}}_r} &:= \overline{\mathbf{D}}_r^*\left[\theta_{\mathbb{C}^m}E_{2\times 2}\right], \\
\mathbf{K}_{\overline{\mathbf{D}}_r^*} &:= \overline{\mathbf{D}}_r\left[\theta_{\mathbb{C}^m}E_{2\times 2}\right].
\end{aligned}$$

Since the matrix $\theta_{\mathbb{C}^m}E_{2\times 2}$ is scalar-valued, one has

$$\mathbf{K}_{\overline{\mathbf{D}}} = \mathbf{K}_{\overline{\mathbf{D}}_r}, \qquad (9.6.3)$$

$$\mathbf{K}_{\overline{\mathbf{D}}^*} = \mathbf{K}_{\overline{\mathbf{D}}_r^*}. \qquad (9.6.4)$$

Using the factorization (9.5.5) we have

$$\overline{\mathbf{D}}\left[\mathbf{K}_{\overline{\mathbf{D}}}\right] = \overline{\mathbf{D}}\circ\overline{\mathbf{D}}^*\left[\theta_{\mathbb{C}^m}E_{2\times 2}\right] = \Delta_{\mathbb{C}^m}\mathbb{E}_{2\times 2}\left[\theta_{\mathbb{C}^m}E_{2\times 2}\right] = \delta_{\mathbb{C}^m}E_{2\times 2}.$$

The same for (9.6.2). By that reason we shall call each of $\mathfrak{W}_m$-valued functions (9.6.1) and (9.6.2) the matrix Cauchy-Dirac kernel for the theory of $\mathfrak{W}_m$-valued functions from the null-set of the corresponding operator.

Note that for every matrix Dirac operator, both the "left-hand-side" theory and the "right-hand-side" theory, have the same matrix Cauchy-Dirac kernel. Compare with Subsection 2.9.

9.7 Borel-Pompeiu formulas for $\mathfrak{W}_m$-valued functions

Theorem *Let Ω^+ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $\mathbf{F} \in C^1(\Omega^+;\ \mathfrak{W}_m) \cap C^0(\Omega^+ \cup \Gamma;\ \mathfrak{W}_m)$. Then the following equalities hold in Ω^+:*

$$\begin{aligned} 2\mathbf{F}(z) &= \int_{\Gamma} \mathbf{K}_{\overline{\mathbf{D}}}(\tau - z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau} \underset{\star}{\wedge} F(\tau) - \\ &\quad - \int_{\Omega^+} \mathbf{K}_{\overline{\mathbf{D}}}(\tau - z) \underset{\star}{\wedge} \overline{\mathbf{D}}[\mathbf{F}](\tau)\, dV_{\tau}, \qquad (9.7.1) \\ 2\mathbf{F}(z) &= \int_{\Gamma} \mathbf{K}_{\overline{\mathbf{D}}^*}(\tau - z) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau} \underset{\star}{\wedge} \mathbf{F}(\tau) - \\ &\quad - \int_{\Omega^+} \mathbf{K}_{\overline{\mathbf{D}}^*}(\tau - z) \underset{\star}{\wedge} \overline{\mathbf{D}}^*[\mathbf{F}](\tau)\, dV_{\tau}, \qquad (9.7.2) \\ 2\mathbf{F}(z) &= \int_{\Gamma} \mathbf{F}(\tau) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau} \underset{\star}{\wedge} \mathbf{K}_{\overline{\mathbf{D}}}(\tau - z) - \\ &\quad - \int_{\Omega^+} \overline{\mathbf{D}}_r[\mathbf{F}](\tau) \underset{\star}{\wedge} \mathbf{K}_{\overline{\mathbf{D}}}(\tau - z)\, dV_{\tau}, \qquad (9.7.3) \\ 2\mathbf{F}(z) &= \int_{\Gamma} \mathbf{F}(\tau) \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}^*_{\tau} \underset{\star}{\wedge} \mathbf{K}_{\overline{\mathbf{D}}^*}(\tau - z) - \\ &\quad - \int_{\Omega^+} \overline{\mathbf{D}}^*_r[\mathbf{F}](\tau) \underset{\star}{\wedge} \mathbf{K}_{\overline{\mathbf{D}}^*}(\tau - z)\, dV_{\tau}. \qquad (9.7.4) \end{aligned}$$

The proof is really the same as that of the analogous theorem in Clifford analysis, see [DeSoSo, Chapter II].

9.8 Monogenic $\mathfrak{W}_m$-valued functions

Definition *Let Ω be an open set in $\mathbb{C}^m$ and $\mathbf{F} \in C^1(\Omega;\, \mathfrak{W}_m)$. Then:*

1. $\mathbf{F}$ is called $\overline{\mathbf{D}}$-monogenic if

$$\overline{\mathbf{D}}\,[\mathbf{F}] = 0, \qquad \text{in } \Omega,$$

$$\overline{\mathfrak{O}}\,(\Omega) := \ker\, \overline{\mathbf{D}};$$

2. $\mathbf{F}$ is called $\overline{\mathbf{D}}^*$-monogenic if

$$\overline{\mathbf{D}}^*\,[\mathbf{F}] = 0, \qquad \text{in } \Omega,$$

$$\overline{\mathfrak{O}}^*\,(\Omega) := \ker\, \overline{\mathbf{D}}^*;$$

3. $\mathbf{F}$ is called $\overline{\mathbf{D}}_r$-monogenic if

$$\overline{\mathbf{D}}_r\,[\mathbf{F}] = 0, \qquad \text{in } \Omega,$$

$$\overline{\mathfrak{O}}_r\,(\Omega) := \ker\, \overline{\mathbf{D}}_r;$$

4. $\mathbf{F}$ is called $\overline{\mathbf{D}}_r^*$-monogenic if

$$\overline{\mathbf{D}}_r^*\,[\mathbf{F}] = 0, \qquad \text{in } \Omega,$$

$$\overline{\mathfrak{O}}_r^*\,(\Omega) := \ker\, \overline{\mathbf{D}}_r^*.$$

9.9 Cauchy's integral representations for monogenic $\mathfrak{W}_m$-valued functions

Theorem *Let Ω^+ be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $\mathbf{F} \in C^1(\Omega^+;\, \mathfrak{W}_m) \cap C^0(\Omega^+ \cup \Gamma;\, \mathfrak{W}_m)$.*

1. *If* $\mathbf{F} \in \overline{\mathfrak{O}}\left(\Omega^{+}\right)$ *then the following equality holds in* Ω^{+}*:*

$$2\mathbf{F}\left(z\right) = \int\limits_{\Gamma} \mathbf{K}_{\overline{\mathbf{D}}}\left(\tau - z\right) \underset{\star}{\cdot} \overline{\boldsymbol{\sigma}}_{\tau} \underset{\star}{\cdot} \mathbf{F}\left(\tau\right). \tag{9.9.1}$$

2. *If* $\mathbf{F} \in \overline{\mathfrak{O}}^{*}\left(\Omega^{+}\right)$ *then the following equality holds in* Ω^{+}*:*

$$2\mathbf{F}\left(z\right) = \int\limits_{\Gamma} \mathbf{K}_{\overline{\mathbf{D}}^{*}}\left(\tau - z\right) \underset{\star}{\cdot} \overline{\boldsymbol{\sigma}}_{\tau}^{*} \underset{\star}{\cdot} \mathbf{F}\left(\tau\right). \tag{9.9.2}$$

3. *If* $\mathbf{F} \in \overline{\mathfrak{O}}_{r}\left(\Omega^{+}\right)$ *then the following equality holds in* Ω^{+}*:*

$$2\mathbf{F}\left(z\right) = \int\limits_{\Gamma} \mathbf{F}\left(\tau\right) \underset{\star}{\cdot} \overline{\boldsymbol{\sigma}}_{\tau} \underset{\star}{\cdot} \mathbf{K}_{\overline{\mathbf{D}}}\left(\tau - z\right). \tag{9.9.3}$$

4. *If* $\mathbf{F} \in \overline{\mathfrak{O}}_{r}^{*}\left(\Omega^{+}\right)$ *then the following equality holds in* Ω^{+}*:*

$$2\mathbf{F}\left(z\right) = \int\limits_{\Gamma} \mathbf{F}\left(\tau\right) \underset{\star}{\cdot} \overline{\boldsymbol{\sigma}}_{\tau}^{*} \underset{\star}{\cdot} \mathbf{K}_{\overline{\mathbf{D}}^{*}}\left(\tau - z\right). \tag{9.9.4}$$

It follows trivially from the Borel-Pompeiu formulas in Subsection 9.7.

9.10 Clifford algebra with the Witt basis and differential forms

We are ready now to imbed elements of $\mathbb{W}_{m}$ into the set of all linear operators acting on differential forms in Ω. First of all we set

$$\begin{aligned}
\varphi\left(\hat{i}^{\mathbf{j}}\right) &:= \widehat{d\bar{z}^{j_1}} \circ \ldots \circ \widehat{d\bar{z}^{j_{|\mathbf{j}|}}} = \widehat{d\bar{z}^{j_1}} \wedge \ldots \wedge \widehat{d\bar{z}^{j_{|\mathbf{j}|}}}, \\
\varphi\left(i^{\mathbf{k}}\right) &:= {}^{d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}}} M; \\
\psi\left(\hat{i}^{\mathbf{j}}\right) &:= \widehat{d\bar{z}^{j_{|\mathbf{j}|}}}_{r} \circ \ldots \circ \widehat{d\bar{z}^{j_1}}_{r}, \\
\psi\left(i^{\mathbf{k}}\right) &:= M^{d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_{|\mathbf{k}|}}};
\end{aligned}$$

where, by definition, for each **d.f.** w on Ω

$$\begin{aligned} {}^{d\bar{z}^{k_1}\wedge\ldots\wedge d\bar{z}^{k_{|\mathbf{k}|}}}M[w] &:= d\bar{z}^{k_1}\wedge\ldots\wedge d\bar{z}^{k_{|\mathbf{k}|}}\wedge w, \\ M^{d\bar{z}^{k_1}\wedge\ldots\wedge d\bar{z}^{k_{|\mathbf{k}|}}}[w] &:= w\wedge d\bar{z}^{k_1}\wedge\ldots\wedge d\bar{z}^{k_{|\mathbf{k}|}}. \end{aligned}$$

Then we extend φ and ψ onto the whole algebra up to an algebra homomorphism:

$$\begin{aligned} \varphi\left(\hat{i}^{\mathbf{j}}\cdot i^{\mathbf{k}}\right) &:= \varphi\left(\hat{i}^{\mathbf{j}}\right)\circ\varphi\left(i^{\mathbf{k}}\right), \\ \psi\left(\hat{i}^{\mathbf{j}}\cdot i^{\mathbf{k}}\right) &:= \psi\left(i^{\mathbf{k}}\right)\circ\psi\left(\hat{i}^{\mathbf{j}}\right), \end{aligned}$$

and for $\mathbf{a}\in\mathbb{W}_m$, $\mathbf{a}=\sum\limits_{\mathbf{jk}}\mathbf{a}_{\mathbf{jk}}\hat{i}^{\mathbf{j}}\wedge i^{\mathbf{k}}$, one has

$$\begin{aligned} \varphi(\mathbf{a}) &:= \sum_{\mathbf{jk}}\mathbf{a}_{\mathbf{jk}}\varphi\left(\hat{i}^{\mathbf{j}}\right)\circ\varphi\left(i^{\mathbf{k}}\right), \\ \psi(\mathbf{a}) &:= \sum_{\mathbf{jk}}\mathbf{a}_{\mathbf{jk}}\psi\left(i^{\mathbf{k}}\right)\circ\psi\left(\hat{i}^{\mathbf{j}}\right). \end{aligned}$$

This definition means that $\varphi:\mathbb{W}_m\to\varphi(\mathbb{W}_m)$ and $\psi:\mathbb{W}_m\to\psi(\mathbb{W}_m)$ are isomorphisms of complex algebras, where the multiplication in $\varphi(\mathbb{W}_m)$ and $\psi(\mathbb{W}_m)$ is the usual composition of operators.

This definition applied pointwise says that each

$$\mathbf{f}=\sum_{\mathbf{jk}}\mathbf{f}_{\mathbf{jk}}\hat{i}^{\mathbf{j}}\cdot i^{\mathbf{k}}\in C^0(\Omega;\,\mathbb{W}_m), \tag{9.10.1}$$

where $\mathbf{f}_{\mathbf{jk}}\in C^0(\Omega;\,\mathbb{C})$ and $\mathbf{j}$, $\mathbf{k}$ are, respectively, strictly increasing $|\mathbf{j}|$-tuple, $|\mathbf{k}|$-tuple, in $\{1,\,\ldots,\,m\}$ with $|\mathbf{j}|$, $|\mathbf{k}|=0,\,\ldots,\,m$, can be identified with the linear operators $\varphi(\mathbf{f})$ and $\psi(\mathbf{f})$ on **d.f.** in Ω defined as follows:

$$\begin{aligned} \varphi(\mathbf{f}) &: w\mapsto\sum_{\mathbf{jk}}\mathbf{f}_{\mathbf{jk}}\otimes\varphi\left(\hat{i}^{\mathbf{j}}\right)\circ\varphi\left(i^{\mathbf{k}}\right)[w], \\ \psi(\mathbf{f}) &: w\mapsto\sum_{\mathbf{jk}}\mathbf{f}_{\mathbf{jk}}\otimes\psi\left(i^{\mathbf{k}}\right)\circ\psi\left(\hat{i}^{\mathbf{j}}\right)[w], \end{aligned}$$

where w is a **d.f.** in Ω and $\otimes$ is the tensor product. In particular, for each $\mathbf{f} \in C^0(\Omega;\ \mathbb{W}_m)$ of the form

$$\mathbf{f} = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} i^{\mathbf{k}},$$

the linear operators $\varphi(\mathbf{f})$ and $\psi(\mathbf{f})$ are, respectively, the operators of multiplication on the left-hand side and the right-hand side, by the following **d.f.** in Ω:

$$f = \sum_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} d\bar{z}^{\mathbf{k}} \in \overline{\mathcal{G}}_0(\Omega).$$

9.11 Relation between the two matrix algebras

Now we extend the mappings φ and ψ onto the matrix algebras with corresponding entries: each $\mathbf{A} \in \mathfrak{W}_m$ can be identified with the two linear operators $\varphi(\mathbf{A})$, $\psi(\mathbf{A})$ on **m.v.d.f.** in Ω, defined by

$$\varphi(\mathbf{A})[F] := \begin{pmatrix} \varphi(\mathbf{A}^{11})[F^{11}] + \varphi(\mathbf{A}^{12})[F^{21}]; & \varphi(\mathbf{A}^{11})[F^{12}] + \varphi(\mathbf{A}^{12})[F^{22}] \\ \varphi(\mathbf{A}^{21})[F^{11}] + \varphi(\mathbf{A}^{22})[F^{21}]; & \varphi(\mathbf{A}^{21})[F^{12}] + \varphi(\mathbf{A}^{22})[F^{22}] \end{pmatrix},$$

$$\psi(\mathbf{A})[F] := \begin{pmatrix} \psi(\mathbf{A}^{11})[F^{11}] + \psi(\mathbf{A}^{21})[F^{12}]; & \psi(\mathbf{A}^{12})[F^{11}] + \psi(\mathbf{A}^{22})[F^{12}] \\ \psi(\mathbf{A}^{11})[F^{21}] + \psi(\mathbf{A}^{21})[F^{22}]; & \psi(\mathbf{A}^{12})[F^{21}] + \psi(\mathbf{A}^{22})[F^{22}] \end{pmatrix}.$$

This definition means that $\varphi : \mathfrak{W}_m \to \varphi(\mathfrak{W}_m)$ and $\psi : \mathfrak{W}_m \to \psi(\mathfrak{W}_m)$ are isomorphisms of complex algebras, where the multiplication in $\varphi(\mathfrak{W}_m)$ and $\psi(\mathfrak{W}_m)$ is the usual composition of operators. This implies, in particular, that for all $\mathbf{A}, \mathbf{B} \in \mathfrak{W}_m$ there hold

$$\varphi\left(\mathbf{A} \underset{\star}{\cdot} \mathbf{B}\right) = \varphi(\mathbf{A}) \underset{\circ}{\cdot} \varphi(\mathbf{B}), \tag{9.11.1}$$

$$\psi\left(\mathbf{A} \underset{\star}{\cdot} \mathbf{B}\right) = \psi(\mathbf{B}) \underset{\circ}{\cdot} \psi(\mathbf{A}). \tag{9.11.2}$$

First of all, note that $C^0(\Omega;\ \mathfrak{W}_m)$ is a complex algebra with the following addition and multiplication. Let $\mathbf{F}, \mathbf{G} \in C^0(\Omega;\ \mathfrak{W}_m)$,

then

$$\begin{aligned}(\mathbf{F}+\mathbf{G})(z) &:= \mathbf{F}(z)+\mathbf{G}(z),\\ \left(\mathbf{F}\underset{\star}{\cdot}\mathbf{G}\right)(z) &:= \mathbf{F}(z)\underset{\star}{\cdot}\mathbf{G}(z).\end{aligned}$$

Each $\mathbf{F}\in C^0(\Omega;\ \mathfrak{W}_m)$ can be identified with the linear operators on **d.f.** in Ω defined as follows:

$$\begin{aligned}\varphi(\mathbf{F})[W](z) &= \varphi(\mathbf{F}(z))[W](z),\\ \psi(\mathbf{F})[W](z) &= \psi(\mathbf{F}(z))[W](z),\end{aligned}$$

where $z\in\Omega$ and W is a **m.v.d.f.** on Ω. Again,

$$\varphi : C^0(\Omega;\ \mathfrak{W}_m)\to\varphi\left(C^0(\Omega;\ \mathfrak{W}_m)\right)$$

and

$$\psi : C^0(\Omega;\ \mathfrak{W}_m)\to\psi\left(C^0(\Omega;\ \mathfrak{W}_m)\right)$$

are isomorphisms of complex algebras, where the multiplication in $\varphi\left(C^0(\Omega;\ \mathfrak{W}_m)\right)$ and $\psi\left(C^0(\Omega;\ \ \mathfrak{W}_m)\right)$ is the usual composition of operators. This implies, in particular, that for all $\mathbf{F},\mathbf{G}\in C^0(\Omega;\ \mathfrak{W}_m)$ there hold

$$\varphi\left(\mathbf{F}\underset{\star}{\cdot}\mathbf{G}\right)=\varphi(\mathbf{F})\underset{\circ}{\wedge}\varphi(\mathbf{G}), \tag{9.11.3}$$

$$\psi\left(\mathbf{F}\underset{\star}{\cdot}\mathbf{G}\right)=\psi(\mathbf{G})\underset{\circ}{\wedge}\psi(\mathbf{F}). \tag{9.11.4}$$

Again, being applied pointwisely, the above says that for each $\mathbf{F}\in C^0(\Omega;\ \mathfrak{W}_m)$ of the form

$$\mathbf{F}:=\sum_{\mathbf{k}}\mathbf{F}_{\mathbf{k}}\begin{pmatrix} i^{\mathbf{k}} & 0\\ 0 & i^{\mathbf{k}}\end{pmatrix}, \tag{9.11.5}$$

where $\mathbf{F}_{\mathbf{k}}\in\overline{\mathfrak{G}}_0^0(\Omega)$, the linear operators $\varphi(\mathbf{F})$ and $\psi(\mathbf{F})$ are, respectively, the operators of multiplication on the left-hand side and on the right-hand side, by the following **m.v.d.f.** on Ω:

$$F:=\sum_{\mathbf{k}}\mathbf{F}_{\mathbf{k}}\begin{pmatrix} d\bar{z}^{\mathbf{k}} & 0\\ 0 & d\bar{z}^{\mathbf{k}}\end{pmatrix}\in\overline{\mathfrak{G}}_0(\Omega).$$

9.11.1 Operators $\overline{\mathcal{D}}$ and $\overline{\mathbf{D}}$

Let Ω be an open set in $\mathbb{C}^m$ and denote by $\overline{\mathbb{G}}_1(\Omega)$ the matrix Grassmann algebra, a subalgebra of $C^1(\Omega;\ \mathfrak{W}_m)$, of all mappings of class C^1 of the form (9.11.5).

Note that the mapping

$$\phi : F = \sum_{\mathbf{k}} F_{\mathbf{k}} \begin{pmatrix} d\bar{z}^{\mathbf{k}} & 0 \\ 0 & d\bar{z}^{\mathbf{k}} \end{pmatrix} \mapsto \phi(F) := \sum_{\mathbf{k}} F_{\mathbf{k}} \begin{pmatrix} i^{\mathbf{k}} & 0 \\ 0 & i^{\mathbf{k}} \end{pmatrix}$$

is an isomorphism between the matrix Grassmann algebra $\overline{\mathfrak{G}}_1(\Omega)$ of all **m.v.d.f.** in Ω of class C^1 and $\overline{\mathbb{G}}_1(\Omega)$.

Theorem *Let Ω be an open set in $\mathbb{C}^m$ and let $F \in \overline{\mathfrak{G}}_1(\Omega)$. Then:*

1. $$\overline{\mathcal{D}}[F] = \varphi\left(\overline{\mathbf{D}}[\phi(F)]\right)[E_{2\times 2}]; \tag{9.11.6}$$

2. *F is hyperholomorphic, if and only if $\varphi\left(\overline{\mathbf{D}}[\phi(F)]\right)[E_{2\times 2}] \equiv 0$.*

Proof. It is sufficient to prove the equality (9.11.6). We have, by definition,

$$\overline{\mathbf{D}}[\phi(F)] = \sum_{p=1}^{m}\sum_{\mathbf{k}} \begin{pmatrix} \frac{\partial F^{11}_{\mathbf{k}}}{\partial \bar{z}_p} i^p \cdot i^{\mathbf{k}} + \frac{\partial F^{21}_{\mathbf{k}}}{\partial z_p} \hat{i}^p \cdot i^{\mathbf{k}}; & \frac{\partial F^{12}_{\mathbf{k}}}{\partial \bar{z}_p} i^p \cdot i^{\mathbf{k}} + \frac{\partial F^{22}_{\mathbf{k}}}{\partial z_p} \hat{i}^p \cdot i^{\mathbf{k}} \\ \frac{\partial F^{11}_{\mathbf{k}}}{\partial z_p} \hat{i}^p \cdot i^{\mathbf{k}} + \frac{\partial F^{21}_{\mathbf{k}}}{\partial \bar{z}_p} i^p \cdot i^{\mathbf{k}}; & \frac{\partial F^{12}_{\mathbf{k}}}{\partial z_p} \hat{i}^p \cdot i^{\mathbf{k}} + \frac{\partial F^{22}_{\mathbf{k}}}{\partial \bar{z}_p} i^p \cdot i^{\mathbf{k}} \end{pmatrix}.$$

Hence

$$\varphi\left(\overline{\mathbf{D}}[\phi(F)]\right) = \sum_{p=1}^{m}\sum_{\mathbf{k}} X_{p,\mathbf{k}},$$

where

$$X_{p,\mathbf{k}} = \left(X^{\alpha,\beta}_{p,\mathbf{k}}\right)_{\alpha,\beta=1}^{2}$$

with

$$X^{11}_{p,\mathbf{k}} = \frac{\partial F^{11}_{\mathbf{k}}}{\partial \bar{z}_p}\, {}^{d\bar{z}^p \wedge d\bar{z}^{\mathbf{k}}}M + \frac{\partial F^{21}_{\mathbf{k}}}{\partial z_p} \widehat{d\bar{z}^p} \circ {}^{d\bar{z}^{\mathbf{k}}}M,$$

$$
\begin{aligned}
X_{p,\mathbf{k}}^{12} &= \frac{\partial F_{\mathbf{k}}^{12}}{\partial \bar{z}_p}\, {}^{d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}}}M + \frac{\partial F_{\mathbf{k}}^{22}}{\partial z_p}\widehat{d\bar{z}^p}\circ {}^{d\bar{z}^{\mathbf{k}}}M,\\
X_{p,\mathbf{k}}^{21} &= \frac{\partial F_{\mathbf{k}}^{11}}{\partial z_p}\widehat{d\bar{z}^p}\circ {}^{d\bar{z}^{\mathbf{k}}}M + \frac{\partial F_{\mathbf{k}}^{21}}{\partial \bar{z}_p}\, {}^{d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}}}M,\\
X_{p,\mathbf{k}}^{22} &= \frac{\partial F_{\mathbf{k}}^{12}}{\partial z_p}\widehat{d\bar{z}^p}\circ {}^{d\bar{z}^{\mathbf{k}}}M + \frac{\partial F_{\mathbf{k}}^{22}}{\partial \bar{z}_p}\, {}^{d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}}}M.
\end{aligned}
$$

Therefore

$$
\varphi\left(\overline{\mathbf{D}}\left[\phi\left(F\right)\right]\right)\left[E_{2\times 2}\right] = \sum_{p=1}^{m}\sum_{\mathbf{k}} Y_{p,\mathbf{k}}
$$

where

$$
Y_{p,\mathbf{k}} = (Y_{p,\mathbf{k}}^{\alpha,\beta})_{\alpha,\beta=1}^{2}
$$

with

$$
\begin{aligned}
Y_{p,\mathbf{k}}^{11} &= \frac{\partial F_{\mathbf{k}}^{11}}{\partial \bar{z}_p}d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}} + \frac{\partial F_{\mathbf{k}}^{21}}{\partial z_p}\widehat{d\bar{z}^p}\left[d\bar{z}^{\mathbf{k}}\right],\\
Y_{p,\mathbf{k}}^{12} &= \frac{\partial F_{\mathbf{k}}^{12}}{\partial \bar{z}_p}d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}} + \frac{\partial F_{\mathbf{k}}^{22}}{\partial z_p}\widehat{d\bar{z}^p}\left[d\bar{z}^{\mathbf{k}}\right],\\
Y_{p,\mathbf{k}}^{21} &= \frac{\partial F_{\mathbf{k}}^{11}}{\partial z_p}\widehat{d\bar{z}^p}\left[d\bar{z}^{\mathbf{k}}\right] + \frac{\partial F_{\mathbf{k}}^{21}}{\partial \bar{z}_p}d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}},\\
Y_{p,\mathbf{k}}^{22} &= \frac{\partial F_{\mathbf{k}}^{12}}{\partial z_p}\widehat{d\bar{z}^p}\left[d\bar{z}^{\mathbf{k}}\right] + \frac{\partial F_{\mathbf{k}}^{22}}{\partial \bar{z}_p}d\bar{z}^p\wedge d\bar{z}^{\mathbf{k}},
\end{aligned}
$$

which means that

$$
\varphi\left(\overline{\mathbf{D}}\left[\phi\left(F\right)\right]\right)\left[E_{2\times 2}\right] = \overline{\mathcal{D}}\left[F\right]. \blacksquare
$$

Remark *Let Ω be an open set in $\mathbb{C}^m$ and $F \in \overline{\mathfrak{G}}_1(\Omega)$.*

1. *Then $\phi(F)$ is $\overline{\mathbf{D}}$-monogenic if, and only if,*

$$
\varphi\left(\overline{\mathbf{D}}\left[\phi\left(F\right)\right]\right)\left[W\right] \equiv 0
$$

for all **m.v.d.f.** *W of class C^0, because φ and ϕ are isomorphisms of algebras.*

2. *The above facts imply that, given F, there are two definitions of its "holomorphy":*

   (a) F is, by definition, $\overline{\mathbf{D}}$-monogenic if $\phi(F)$ is $\overline{\mathbf{D}}$-monogenic;

   (b) F is, by definition, $\overline{\mathcal{D}}$-hyperholomorphic if $\overline{\mathcal{D}}[F] \equiv 0$.

Let us consider some consequences of the first of these definitions. Hyperholomorphic theory treats **m.v.d.f.**

$$F := \sum_{\mathbf{k}} \mathbf{F}_{\mathbf{k}} \begin{pmatrix} d\bar{z}^{\mathbf{k}} & 0 \\ 0 & d\bar{z}^{\mathbf{k}} \end{pmatrix}$$

which behave, algebraically, like elements of a Grassmann subalgebra of the Clifford algebra, i.e., like the Grassmann algebra

$$\overline{\mathbb{G}}_1(\Omega) = \left\{ \sum_{\mathbf{k}} F_{\mathbf{k}} \begin{pmatrix} i^{\mathbf{k}} & 0 \\ 0 & i^{\mathbf{k}} \end{pmatrix} \right\},$$

and the isomorphism

$$\phi : F = \sum_{\mathbf{k}} \mathbf{F}_{\mathbf{k}} \begin{pmatrix} d\bar{z}^{\mathbf{k}} & 0 \\ 0 & d\bar{z}^{\mathbf{k}} \end{pmatrix} \mapsto \phi(F) : \sum_{\mathbf{k}} F_{\mathbf{k}} \begin{pmatrix} i^{\mathbf{k}} & 0 \\ 0 & i^{\mathbf{k}} \end{pmatrix}$$

establishes a way to identify them. But on $\overline{\mathbb{G}}_1(\Omega)$ we know already what it means to be monogenic, and this leads to that first definition. Unfortunately, this is not an interesting situation for us. Indeed, take a function $f : \Omega \to \mathbb{C}$, considering it as a **d.f.** Then $\phi(fE_{2\times 2}) = fE_{2\times 2}$, but now the same expression is seen as a $\mathfrak{W}_m$-valued function. Moreover,

$$\overline{\mathbf{D}}\left[\phi\left(fE_{2\times 2}\right)\right] = \sum_{j=1}^{m} \begin{pmatrix} \frac{\partial f}{\partial \bar{z}_j} i^j; & \frac{\partial f}{\partial z_j} \hat{i}^j \\ \\ \frac{\partial f}{\partial z_j} \hat{i}^j; & \frac{\partial f}{\partial \bar{z}_j} i^j \end{pmatrix};$$

hence, $fE_{2\times 2}$ is $\overline{\mathbf{D}}$ - monogenic if and only if

$$\frac{\partial f}{\partial \bar{z}_j} = 0,$$

$$\frac{\partial f}{\partial z_j} \quad = \quad 0,$$

for all $j = 1, \ldots, m$; that is,

$$f \equiv \text{ constant.}$$

In other words, among all the functions, only constant functions generate monogenic elements in $\overline{\mathbb{G}}_1$.

Thus the "right" approach is the second one, which implies a theory with the structure not coinciding with the structure of Clifford analysis, but it contains all holomorphic functions in the sense of several complex variables, which are the object of our study. One of the principal reasons why hyperholomorphic theory is different from Clifford analysis is contained in the asymmetry of the following formula:

$$\varphi\left(\mathbf{A} \underset{\star}{\cdot} \mathbf{B}\right)[W] = \langle \varphi(\mathbf{A})[W],\ \varphi(\mathbf{B}) \rangle,$$

where $\langle\ ,\ \rangle$ denotes the evaluation operator and W is an arbitrary **m.v.d.f.**

One more comment to the formula (9.11.6). Denote by $\mathbb{E}$ the evaluation mapping defined on operators which act on **m.v.d.f.**, by the formula: if u is such an operator then

$$\mathbb{E}[u] := u[E_{2\times 2}].$$

The formula (9.11.6) gives

$$\overline{\mathcal{D}} \ = \ \mathbb{E} \circ \varphi \circ \overline{\mathbf{D}} \circ \phi,$$

where φ and ϕ are isomorphisms, but $\mathbb{E}$ is not, hence $\ker\overline{\mathcal{D}}$ and $\ker\overline{\mathbf{D}}$ are essentially different.

Lemma. *For all $\zeta \in \mathbb{C}^m$ there holds*

$$\mathcal{K}_{\overline{\mathcal{D}}}(\zeta,\ z) \quad = \quad \varphi\left(\mathbf{K}_{\overline{\mathbf{D}}}(\zeta)\right), \tag{9.11.7}$$

Proof. This lemma is a direct implication of the definition of φ. ■

This justifies completely the definition of $\mathcal{K}_{\overline{\mathcal{D}}}$ and, in this sense, it is a "fundamental solution" of the corresponding operator $\overline{\mathcal{D}}$.

9.12 Cauchy's integral representation for left-hyperholomorphic matrix-valued differential forms

Theorem *Let Ω^{+} be a bounded domain with the topological boundary Γ, which is a piecewise smooth surface, let $F \in \overline{\mathfrak{N}}\left(\Omega^{+}\right) \cap \overline{\mathfrak{G}}_{0}\left(\Omega^{+} \cup \Gamma\right)$. Then the following equality in Ω^{+},*

$$F(z)=\int_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}_{\tau-z,\, z}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge} F(\tau,\, d\bar{z}), \tag{9.12.1}$$

is a corollary of Borel-Pompeiu's integral representation from Subsection 9.7.

Proof. First, note that since $F \in \overline{\mathfrak{N}}\left(\Omega^{+}\right) \cap \overline{\mathfrak{G}}_{0}\left(\Omega^{+} \cup \Gamma\right)$ then $\phi(F) \in C^{1}\left(\Omega^{+};\, \mathfrak{W}_{m}\right) \cap C^{0}\left(\Omega^{+} \cup \Gamma;\, \mathfrak{W}_{m}\right)$. So, by Theorem 9.7 we have

$$\begin{aligned} 2\phi(F)(z) &= \int_{\Gamma} \mathbf{K}_{\overline{\mathbf{D}}}(\tau-z) \underset{\star}{\cdot} \overline{\boldsymbol{\sigma}}_{\tau} \underset{\star}{\cdot} \phi(F)(\tau) - \\ &\quad - \int_{\Omega^{+}} \mathbf{K}_{\overline{\mathbf{D}}}(\tau-z) \underset{\star}{\cdot} \overline{\mathbf{D}}[\phi(F)](\tau)\, dV_{\tau}. \end{aligned}$$

Hence

$$\begin{aligned} 2\varphi(\phi(F)(z)) &= \int_{\Gamma} \varphi\left(\mathbf{K}_{\overline{\mathbf{D}}}(\tau-z)\right) \underset{\star}{\wedge} \varphi\left(\overline{\boldsymbol{\sigma}}_{\tau}\right) \underset{\star}{\wedge} \varphi(\phi(F)(\tau)) - \\ &\quad - \int_{\Omega^{+}} \varphi\left(\mathbf{K}_{\overline{\mathbf{D}}}(\tau-z)\right) \underset{\star}{\wedge} \varphi\left(\overline{\mathbf{D}}[\phi(F)](\tau)\right) dV_{\tau} = \\ &= \int_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}_{\tau-z,\, z}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge}\, {}^{F(\tau)}M - \\ &\quad - \int_{\Omega^{+}} \mathcal{K}_{\overline{\mathcal{D}}}(\tau-z,\, z) \underset{\star}{\wedge} \varphi\left(\overline{\mathbf{D}}[\phi(F)](\tau)\right) dV_{\tau}. \end{aligned}$$

Therefore

$$2\, {}^{F(z)}M\left[E_{2\times 2}\right] = \int_{\Gamma} \mathcal{K}_{\overline{\mathcal{D}}_{\tau-z,\, z}} \underset{\star}{\wedge} \overline{\boldsymbol{\sigma}}_{\tau,\, z} \underset{\star}{\wedge}\, {}^{F(\tau)}M\left[E_{2\times 2}\right] -$$

$$-\int\limits_{\Omega^+}\mathcal{K}_{\overline{\mathcal{D}}}\left(\tau-z,\ z\right)\underset{\star}{\wedge}\varphi\left(\overline{\mathbf{D}}\left[\phi\left(F\right)\right]\left(\tau\right)\right)\left[E_{2\times 2}\right]dV_{\tau}=$$

$$=\int\limits_{\Gamma}\mathcal{K}_{\overline{\mathcal{D}}\tau-z,\ z}\underset{\star}{\wedge}\overline{\boldsymbol{\sigma}}_{\tau,\ z}\underset{\star}{\wedge}F\left(\tau,d\bar{z}\right)-$$

$$-\int\limits_{\Omega^+}\mathcal{K}_{\overline{\mathcal{D}}}\left(\tau-z,\ z\right)\underset{\star}{\wedge}\overline{\mathcal{D}}\left[F\right]\left(\tau,\ d\bar{z}\right)dV_{\tau}.$$

Now, as $F\in\overline{\mathfrak{N}}\left(\Omega^+\right)$ implies that $\overline{\mathcal{D}}\left[F\right]\left(\tau\right)=0$ for all τ, we have

$$2F\left(z\right)=\int\limits_{\Gamma}\mathcal{K}_{\overline{\mathcal{D}}\tau-z,\ z}\underset{\star}{\wedge}\overline{\boldsymbol{\sigma}}_{\tau,\ z}\underset{\star}{\wedge}F\left(\tau,\ d\bar{z}\right).\blacksquare$$

Note that it is impossible to use the Cauchy integral representation from Subsection 9.9 to prove this theorem, because $F\in\overline{\mathfrak{N}}\left(\Omega^+\right)$ implies only that $\varphi\left(\overline{\mathbf{D}}\left[\phi\left(F\right)\right]\left(\tau\right)\right)\left[E_{2\times 2}\right]\equiv 0$ for all τ, and in general $\overline{\mathbf{D}}\left[\phi\left(F\right)\right]\neq 0$.

9.13 Hyperholomorphic theory and Clifford analysis

Section 9.3 says, in fact, that on matrices of the form $\begin{pmatrix} f & g \\ f & g \end{pmatrix}$ the hyperholomophic theory reduces to just Clifford analysis in $\mathbb{R}^{2m}$ endowed with the complex structure. Now we are going to find out what the relation is between the hyperholomophic theory for arbitrary matrices and Clifford analysis. First of all, consider $\overline{\boldsymbol{\partial}}-\overline{\boldsymbol{\partial}}^{\wedge}$. We have

$$\begin{aligned}\overline{\boldsymbol{\partial}}-\overline{\boldsymbol{\partial}}^{\wedge} &= \sum_{k=1}^{m}i^k\frac{\partial}{\partial\overline{z}_k}-\sum_{k=1}^{m}i^k\frac{\partial}{\partial z_k}= \\ &= \frac{1}{2}\sum_{k=1}^{m}i^k\left(\frac{\partial}{\partial x_k}+i\frac{\partial}{\partial y_k}\right)- \\ &\quad -\frac{1}{2}\sum_{k=1}^{m}\widehat{i^k}\left(\frac{\partial}{\partial x_k}-i\frac{\partial}{\partial y_k}\right)=\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{m}\left(i^k - \widehat{i}^k\right)\frac{\partial}{\partial x_k} + \frac{1}{2}i\sum_{k=1}^{m}\left(i^k + \widehat{i}^k\right)\frac{\partial}{\partial y_k} = \\
&= \frac{1}{2}\left(\sum_{k=1}^{m} e_{2k}\frac{\partial}{\partial x_k} - \sum_{k=1}^{m} e_{2k-1}\frac{\partial}{\partial y_k}\right) = \\
&= -\frac{1}{2}\left(\sum_{k=1}^{m} e_{2k-1}\frac{\partial}{\partial y_k} - \sum_{k=1}^{m} e_{2k}\frac{\partial}{\partial x_k}\right) .
\end{aligned}
$$

Setting

$$
\vartheta_t^+ := \vartheta_t \tag{9.13.1}
$$

and

$$
\vartheta_t^- := \sum_{k=1}\left(e_{2k-1}\frac{\partial}{\partial y_k} - e_{2k}\frac{\partial}{\partial x_k}\right) , \tag{9.13.2}
$$

we get

$$
\vartheta_t^+ = -2i\left(\overline{\boldsymbol{\partial}} + \overline{\boldsymbol{\partial}}^{\wedge}\right) , \tag{9.13.3}
$$

$$
\vartheta_t^- = -2\left(\overline{\boldsymbol{\partial}} - \overline{\boldsymbol{\partial}}^{\wedge}\right) . \tag{9.13.4}
$$

Note that is follows directly that

$$
\begin{aligned}
\overline{\boldsymbol{\partial}} &= \frac{i}{4}\left(\vartheta_t^+ + i\vartheta_t^-\right) ; \\
\overline{\boldsymbol{\partial}}^{\wedge} &= \frac{i}{4}\left(\vartheta_t^+ - i\vartheta_t^-\right) .
\end{aligned} \tag{9.13.5}
$$

Consider now the matrix Dirac operator

$$
\overline{\mathbf{D}} := \begin{pmatrix} \overline{\boldsymbol{\partial}} & \overline{\boldsymbol{\partial}}^* \\ \overline{\boldsymbol{\partial}}^* & \overline{\boldsymbol{\partial}} \end{pmatrix}
$$

and the operators

$$
\mathbb{E}_{2\times 2} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \breve{\mathbb{E}}_{2\times 2} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
$$

introduced in Subsection 2.3. Since $\breve{\mathbb{E}}_{2\times 2}$ is an involution, $\breve{\mathbb{E}}_{2\times 2}^2 = \mathbb{E}_{2\times 2}$, it generates a pair of mutually complementary projections

$$
\mathbb{E}_{\pm} := \frac{1}{2}\left(\mathbb{E}_{2\times 2} \pm \breve{\mathbb{E}}_{2\times 2}\right) .
$$

More explicitly,

$$\mathbb{E}_+ = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad \mathbb{E}_- = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

with

$$\mathbb{E}_\pm^2 = \mathbb{E}_\pm, \quad \mathbb{E}_+ - \mathbb{E}_- = \breve{\mathbb{E}}_{2\times 2};$$

$$\mathbb{E}_+ + \mathbb{E}_- = \mathbb{E}_{2\times 2}; \quad \mathbb{E}_+ \circ \mathbb{E}_- = \mathbb{E}_- \circ \mathbb{E}_+ = 0\,.$$

For the matrix Dirac operator $\overline{\mathbf{D}}$ this gives:

$$\begin{aligned} \overline{\mathbf{D}} &= \overline{\boldsymbol{\partial}} \cdot \mathbb{E}_{2\times 2} + \overline{\boldsymbol{\partial}}^{\wedge} \cdot \breve{\mathbb{E}}_{2\times 2} = \\ &= \overline{\boldsymbol{\partial}} \cdot (\mathbb{E}_+ + \mathbb{E}_-) + \overline{\boldsymbol{\partial}}^{\wedge} \cdot (\mathbb{E}_+ - \mathbb{E}_-) = \\ &= \left(\overline{\boldsymbol{\partial}} + \overline{\boldsymbol{\partial}}^{\wedge}\right) \cdot \mathbb{E}_+ + \left(\overline{\boldsymbol{\partial}} - \overline{\boldsymbol{\partial}}^{\wedge}\right) \cdot \mathbb{E}_- = \\ &= \frac{i}{2}\vartheta_t^+ \cdot \mathbb{E}_+ - \frac{1}{2}\vartheta_t^- \cdot \mathbb{E}_- \quad . \end{aligned} \tag{9.13.6}$$

Let $\mathbf{F} = (\mathbf{F}_{kj})_{k,j=1}^2$ be a smooth $\mathfrak{W}_m$-valued function, then it decomposes in a unique way into

$$\mathbf{F} = \mathbf{F}_+ - \mathbf{F}_- \tag{9.13.7}$$

where

$$\mathbf{F}_+ := \mathbb{E}_+ [\mathbf{F}] = \frac{1}{2}\begin{pmatrix} \mathbf{F}_{11} + \mathbf{F}_{21}\,; & \mathbf{F}_{12} + \mathbf{F}_{22} \\ \mathbf{F}_{11} + \mathbf{F}_{21}\,; & \mathbf{F}_{12} + \mathbf{F}_{22} \end{pmatrix},$$

$$\mathbf{F}_- := \mathbb{E}_- [\mathbf{F}] = \frac{1}{2}\begin{pmatrix} \mathbf{F}_{11} - \mathbf{F}_{21}\,; & \mathbf{F}_{12} - \mathbf{F}_{22} \\ -\mathbf{F}_{11} + \mathbf{F}_{21}\,; & -\mathbf{F}_{12} + \mathbf{F}_{22} \end{pmatrix}.$$

The definition of a $\overline{\mathbf{D}}$-monogenic function, $\overline{\mathbf{D}}[\mathbf{F}] = 0$, turns out to be

$$i\vartheta_t^+ \mathbb{E}_+ [\mathbf{F}] - \vartheta_t^- \mathbb{E}_- [\mathbf{F}] = 0\,,$$

or, equivalently,

$$i\vartheta_t^+ \mathbb{E}_+ \left[\mathbf{F}_+\right] - \vartheta_t^- \mathbb{E}_- \left[\mathbf{F}_-\right] = 0 \, . \tag{9.13.8}$$

The last equality is valid if and only if each of the projections of the function $\mathbf{F}$, respectively $\mathbf{F}_+$ and $\mathbf{F}_-$, belongs to the class of functions monogenic (synonymically, regular, hyperholomorphic, etc.) in the sense of Clifford analysis. The fine point here is that while $\mathbf{F}_+$ is monogenic in the sense of the "canonical" Clifford analysis in the domain Ω, of that one which is determined by ϑ_t^+, its counterpart $\mathbf{F}_-$ is monogenic, in the same domain, in the sense of the operator ϑ_t^-. Of course, if one is interested in only one of the two versions, then there is no essential difference between them. But when, as the equality (9.13.6) shows, we are looking to "glue together" **both of the two versions,** then the situation is not that simple. What is more, the equality (9.13.6) shows that, at least in principle, it is possible to obtain the main facts of the hyperholomorphic theory of matrix-valued differential forms as a "direct sum" of the above two versions of Clifford analysis. For many reasons, we chose the direct way of constructing.

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